ON VARIETIES OF SEMIGROUPS

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AFTAB HUSSAIN SHAH

Under the Supervision of

Dr. NOOR MOHAMMAD KHAN

DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH (INDIA)

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Dedicated

To My

Parents

DR. NOOR MOHAMMAD KHAN
Reader

DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH – 202002 (INDIA)
E-mail: khan_noormohammad@rediffmail.com

Certificate

This is to certify that the dissertation entitled "***On Varieties of Semigroups***" has been written by ***Mr. Aftab Hussain Shah*** under my supervision for submission in Aligarh Muslim University, Aligarh as a partial fulfillment for the award of the degree of Master of Philosophy in Mathematics. To the best of my knowledge, the exposition has not been submitted to any other University or institution for the award of a degree/diploma and is suitable for submission in all respects.

N.M. Khan

Dr. Noor Mohamamd Khan
(Supervisor)

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(AFTAB HUSSAIN SHAH)

PREFACE

The present dissertation entitled **On Varieties of Semigroups** studies the following question. Which Permutative varieties of semigroups are closed under Epis?

In the present dissertation we study the above question in the context of the semigroup varieties by considering the following two questions.

Q(a): What are the permutative saturated varieties of semigroups?

Q(b): What are epimorphically closed permutative varieties of semigroups?

Clearly saturated varieties of semigroups are epimorphically closed.

The present exposition consists of five chapters. Chapter 1 contains preliminary notions, basic definitions and some fundamental results including the celebrated Zigzag theorem (with proof) which are used to develop the theory in the subsequent chapters.

The Chapter 2 opens with the definitions of the saturated and epimorphically closed varieties of semigroups and is divided into two sections. In the first section we give a necessary condition for a semigroup to be saturated by constructing an example and, then, prove certain results to be used to completely answer Q(a). In the next section we give a necessary condition for a semigroup variety, again by constructing an example, to be epimorphically closed.

In Chapter 3, we extend the result that "commutativity is preserved under epis" to permutation identities and proved that all permutation identities are preserved under epis.

In Chapter 4, after proving some consequences of permutation identities about permutative semigroups, we completely determine permutative saturated varieties of semigroups. We then find sufficient conditions on a homotypical identity to ensure that any variety satisfying it is saturated.

In Chapter 5, we answer the Q(b), i.e. we ask the following question: which permutation identities are stable? or equivalently which permutative varieties are closed under epis?.

First we give some sufficient conditions on semigroup identities to be preserved under epis in conjunction with any nontrivial permutation identity and, then, we completely answer the above question and show that

$$x_1x_2...x_n = x_{i_1}x_{i_2}...x_{i_n}$$

$n \geq 3$ is stable if and only if $i_n \neq n[i_1 \neq 1]$.

At the end, the list of references of literature consulted has been given.

CHAPTER 1

INTRODUCTION

This chapter is devoted to collect some basic semigroup theoretic notions and results with a view to make our dissertation as self contained as possible, whereas the elementary knowledge of the algebraic concepts such as groups, homomorphisms, monomorphisms etc. has been preasumed and, thus, no attempt is been made to discuss them here. Most of the material included in this chapter occurs in the standard literature namely [6], [7], [38], [39] and [40].

1.1 BASIC DEFINITIONS

In this section we begin with some basic definitions of semigroup theory.

DEFINITION 1.1.1: A non-empty set S with a binary operation $\star$ which is associative is called a *semigroup.*

$a \star b$ for a, $b \in S$ will be denoted by ab. We shall write a multiplicative semigroup as $(S, \cdot)$ or often simply as S.

DEFINITION 1.1.2: If a semigroup $(S, \cdot)$ has the additional property that (for all $x, y \in S$).

$$xy = yx,$$

we shall say that it is a *commutative semigroup.*

DEFINITION 1.1.3: Let S be a semigroup. An element a of S is regular if there exists $x \in S$ such that $a = axa$. A semigroup S is said to be *regular* if all its elements are regular.

DEFINITION 1.1.4: A semigroup S is called a *union of groups* if each of its elements is contained in some subgroup of S. If a is an element of such a semigroup then $a \in G$, a subgroup of S. If we denote the identity of G by e then within the group G, we have

$$ea = ae = a, \quad aa^{-1} = a^{-1}a = e;$$

DEFINITION 1.1.5(BAND): By a *band* we mean a semigroup S in which every element is idempotent, i.e $a^2 = a$ for all $a \in S$.

DEFINITION 1.1.6: If I and Λ are non-empty sets, an associative multiplication can be defined on the Cartesian product.

$$I \times \Lambda \;=\; \{(\lambda, \mu) : \;\; \lambda \in I, \;\; \mu \in \Lambda\}$$

as follows

$$(\lambda_1, \mu_1) \star (\lambda_2, \mu_2) \;=\; (\lambda_1, \mu_2).$$

Then $(I \times \Lambda, \star)$ is a semigroup which is called a *rectangular band.* If $|\Lambda| = 1$ $[|I| = 1]$, then the rectangular band $I \times \Lambda$ is a left zero [right zero] semigroup.

DEFINITION 1.1.7: A band S is said to be *right [left] normal band* if $abc = bac$ $[abc = acb]$, for all $a, b, c \in S$.

DEFINITION 1.1.8: A semigroup S is called an *inverse semigroup* if every a in S possesses a unique inverse, i.e. if there exists a unique element a^{-1} in S such that

$$aa^{-1}a \;=\; a, \;\; a^{-1}aa^{-1} \;=\; a^{-1}$$

Such a semigroup is certainly regular, but not every regular semigroup is an inverse semigroup; a rectangular band is an obvious example as every element is an inverse of every other element.

DEFINITION 1.1.9: If S and T are semigroups, the Cartesian product $S \times T$ becomes a semigroup if we define

$$(s, t)(s', t') \;=\; (ss', tt')$$

we shall refer to this semigroup as the direct product of S and T.

DEFINITION 1.1.10: A relation ρ on a set X is called an *equivalence relation* if it is reflexive, symmetric and transitive. If ρ is an equivalence relation on X then

$$\mathrm{dom}(\rho) \supseteq \mathrm{dom}(1_x) \;=\; X, \;\; \mathrm{ran}(\rho) \supseteq \mathrm{ran}(1_x) \;=\; X$$

and so $\mathrm{dom}(\rho) = \mathrm{ran}(\rho) = X$.

DEFINITION 1.1.11: Let R be a relation on a semigroup S. If $c, d \in S$ are such that

$$c \;=\; xpy, \;\; d \;=\; xqy$$

for some x, y in S^1, where either $(p, q) \in R$ or $(q, p) \in R$, we say that c is connected to d by an *elementary R-transition*, where

$$S^1 = \begin{cases} S \text{ if } S \text{ has } 1 \\ S \cup \{1\}, \quad \text{otherwise} \end{cases}$$

DEFINITION 1.1.12: A semigroup S is *right [left] simple* if $aS = S$ $[Sa = S]$ for all $a \in S$. A global definition of a group, often utilized in semigroup theory, is that of a semigroup which is both left and right simple. One might therefore accept right simple semigroups to resemble with groups. The classical result along these lines is that the following are equivalent:

(i) S is right simple and left cancellative;

(ii) S is right simple with at least one idempotent;

(iii) S is isomorphic to $G \times R$, where G is a group and R is a right zero-semigroup (i.e. $xy = y$ for all $x, y \in R$).

This characterization of these semigroups, known as right groups is given in Clifford and Preston [6].

DEFINITION 1.1.13: If a is an element of a semigroup S, the smallest left [right] ideal containing a is $Sa \cup \{a\}$ $[\{a\} \cup aS]$ denoted by S^1a $[aS^1]$, and which we call principal left [right] ideal generated by a.

Green's relation $\mathcal{L}$ on S $[\mathcal{R}$ on $S]$ is then defined by, that $a\mathcal{L}b \Leftrightarrow S^1a = S^1b$

$$\Leftrightarrow \exists x, \quad y \in S^1 \text{ such that } a = xb \text{ and } b = ya.$$

Dually $a\mathcal{R}b \Leftrightarrow aS^1 = bS^1$

$$\Leftrightarrow \exists x, \quad y \in S^1 \text{ such that } a = bx \text{ and } b = ay.$$

The join of the Green's relations $\mathcal{L}$ and $\mathcal{R}$ in the lattice of all equivalences on a semigroup is called the Green's relation $\mathcal{D}$

DEFINITION 1.1.14: Each $\mathcal{D}$ class in a semigroup is a union of $\mathcal{L}$-classes and also a union of $\mathcal{R}$-classes. The intersection of $\mathcal{L}$-class and $\mathcal{R}$-class is either empty or is an

$\mathcal{H}$-class (the Green's relation $\mathcal{H}$ is defined as $\mathcal{L} \cap \mathcal{R}$). In fact, by the very definition of $\mathcal{D}$

$$a \,\mathcal{L}\, b \Leftrightarrow R_a \cap L_b \neq \emptyset$$
$$\Leftrightarrow L_a \cap R_b \neq \emptyset$$

where $L_a[R_a]$, for any $a \in S$ is the $\mathcal{L}[\mathcal{R}]$ class of a.

DEFINITION 1.1.15: A semigroup S is *globally idempotent* if for every $a \in S$ there exist $x, y \in S$ such that $a = xy$, that is, if $S = S^2$. For example every regular semigroup is globally idempotent.

1.2 SOME SELECTED RESULTS

In this section after the preliminary definitions we have given some important theorems and results to be used in subsequent chapters including the celebrated zigzag theorem due to Isbell. The importance of this theorem to the present dissertation has prompted us to include a full proof of this theorem, although the proof can be found in the introductory text by Howie [38].

DEFINITION 1.2.1(VARIETIES): Let C be a class of algebras, all of the same type (in the sense of Gratzer [19]). Let F be the set of operations defined on algebra in C. Let $\{x_i : i \in N\}$ be a countable set (where N denotes the set of all positive integers). A word in the variables x_i $(i \in N)$ is defined as follows:

(a) every variable x_i $(i \in N)$ is a word;

(b) if α is the symbol of a nullary operation from F than α is a word;

(c) if $y_1, y_2, \ldots, y_n$ are words and f is an n-ary operation from F, then $f(y_1, y_2, \ldots, y_n)$ is also a word;

(d) words are those and only those objects which we get from (a), (b) and (c) in a finite number of steps.

Let C, F and $\{x_i : i \in N\}$ be as above. An identity is a pair (u, v) where u and v are words in the variables x_i $(i \in N)$. The identity (u, v) is usually written as

$$u = v.$$

Often, we use variables other than x_i $(i \in N)$; for example, the identity $x_1x_2 = x_2x_1$ will be written as $xy = yx$.

We say that an algebra A from C satisfies the identity $u = v$ if for every substitution of the variables in u and v by elements of A, the resulting elements in A are equal.

A set of identities $\{u_i = v_i : i \in I\}$ implies a set of identities $\{u'_j = v'_j : j \in J\}$ if every algebra in C satisfying all $u_i = v_i$ also satisfies $u'_j = v'_j$ and two sets of identities are equivalent if the converse also holds.

The class of algebras of a certain fixed type satisfying a given set of identities $\{u_i = v_i : i \in I\}$ is called the variety determined by the set of identities $\{u_i = v_i : i \in I\}$.

The following theorem is due to G. Birkhoff.

THEOREM 1.2.2[10, Theorem 3.1 Chapter IV]: Let C be a class of algebras, all of the same type. Then C is a variety if and only if C is closed under the taking of subalgebras, direct products and homomorphic images.

Let $\{A_i : i \in I\}$ be a family of algebras, all of the same type. Let B be the direct product of the family, and let A be a subalgebra of B. Then A is called a sub-direct product of the family $\{A_i : i \in I\}$ if the projections from A to the A_i $(i \in I)$ are surjective

THEOREM 1.2.3[10, Corollary 2.6 Chapter IV]: Let C be a class of algebras, all of the same type. Then C is a variety if and only if C is closed under the taking of subdirect products and homomorphic images.

Let u be any word. The *content* of u is the (necessarily finite) set of all distinct variables appearing in u, and will be denoted by $C(u)$. Further for any variable x of u, $|x|_u$ will denote the number of occurrence of the variable x in the word u.

An identity $u(x_1, x_2, \ldots, x_n) = v(x_1, x_2, \ldots, x_n)$ is called homotypical if $C(u) = C(v)$ and heterotypical if $C(u) \neq C(v)$.

By a permutation identity in the variables $x_1, x_2, \ldots, x_n$ $(n \geq 2)$ we mean an identity

$$x_1x_2 \ldots x_n = x_{i_1}x_{i_2} \ldots x_{i_n} \tag{1}$$

where $(i_1, i_2, \ldots, i_n)$ is a permutation of the sequence $(1, 2, \ldots, n)$.

The identities

(C) commutativity, $xy = yx$,

(LN) left normality, $x_1x_2x_3 = x_1x_3x_2$,

(RN) right normality, $x_1x_2x_3 = x_2x_1x_3$,

(N) normality, $x_1x_2x_3x_4 = x_1x_3x_2x_4$,

are all permutation identities.

The identity (1) is said to be *nontrivial* if the permutation $(i_1, i_2, \ldots, i_n)$ of the sequence $(1, 2, \ldots, n)$ is different from the identity permutation.

DEFINITION 1.2.4(EPIMORPHISMS AND DOMINIONS): Let C be the category of all semigroups and let U be a subsemigroup of a semigroup S in C. We say that U dominates an element d of S if for every semigroup $T \in C$ and for all homomorphisms $\beta, \gamma : S \to T$, such that $u\beta = u\gamma$ for all $u \in U$ implies $d\beta = d\gamma$. The set of all elements dominated by U is called the dominion of U in S and we denote it by $\mathrm{Dom}_C(U, S)$. It can be easily seen that $\mathrm{Dom}_C(U, S)$ is a subsemigroup of S containing U. In what follows,if C the category of all semigroups, $\mathrm{Dom}_C(U, S)$ will be denoted by $\mathrm{Dom}(U, S)$.

RESULT 1.2.5:[42, Theorem 2.3] Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in \mathrm{Dom}(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:

$$d = u_0y_1 = x_1u_1y_1 = x_1u_2y_2 = x_2u_3y_2 = \ldots = x_mu_{2m-1}y_m = x_mu_{2m}$$

where $m \geq 1$, $u_i \in U$, $x_i, y_i \in S$ with $u_0 = x_1u_1, u_{2m-1}y_m = u_{2m}$

$$x_iu_{2i} = x_{i+1}u_{2i+1}, u_{2i-1}y_i = u_{2i}y_{i+1}, (1 \leq i \leq m-1) \qquad (2)$$

Such a series of factorization is called a zigzag in S over U with value d, length m and spine $u_0, u_1, \ldots, u_{2m}$.

This result is so important for our purposes that we include a full proof, although this proof can also be found in the introductory text by Howie [38].

The proof in the reverse direction is just a straight forward zigzag manipulation. Suppose z is a zigzag with value d in S over U and that $\alpha, \beta : S \to T$ are two semigroup morphisms such that $\alpha|U = \beta|U$. Then

$$\begin{aligned} d\alpha &= (u_0y_1)\alpha = (u_0\alpha)(y_1\alpha) = (u_0\beta)(y_1\alpha) = \{(x_1u_1)\beta\}y_1\alpha = (x_1\beta)(u_1\beta)(y_1\alpha) \\ &= x_1\beta(u_1\alpha)y_1\alpha = x_1\beta(u_1y_1)\alpha = x_1\beta(u_2y_2)\alpha = \ldots = x_m\beta(u_{2m-1}y_m)\alpha \\ &= x_m\beta(u_{2m}\alpha) = x_m\beta(u_{2m}\beta) = (x_mu_{2m})\beta = d\beta \end{aligned}$$

as required.

The proof of the converse part is more formidable and is momentarily delayed. To give the reader a little more feeling for zigzag manipulation we include the following surprising fact.

PROPOSITION 1.2.6(Howie and Isbell [40]): Let z be a zigzag in S over U with value d and spine $u_0, u_1, \ldots, u_{2m}$. Let z' be a zigzag in S over U with the same spine. Then the value of z' is also d.

Two such zigzags are therefore called equivalent.

PROOF: Suppose z is given by $u_0y_1 = x_1u_1y_1 = \ldots = x_mu_{2m}$, while z' by $u_0y_1' = x_1'u_1y_1' = \ldots = x_m'u_{2m}$, with the appropriate zigzag equalities, as given by Result 1.2.5 holding for both. Then we have $d = u_0y_1 = x_1'u_1y_1 = x_1'u_2y_2 = x_2'u_3y_2 = x_2'u_4y_3 = x_3'u_5y_3 = \ldots = x_m'u_{2m-1}y_m = x_m'u_{2m} =$ the value of z', as required.

We now prepare for a proof of the zigzag theorem. If M is a set and S is a semigroup with identity 1, we say that M is a *right S-system* if there is a mapping $(x, s) \to xs$ from $M \times S$ into S such that $(xs)t = x(st)$ $(x \in M,\ s, t \in S)$ and $x1 = x$ $(x \in M)$. A left S-system is defined dually. If S and T are semigroup with identity, we say that M is an (S, T)-bisystem if it is *left* S-system, a right T-system and if for all $s \in S$, $t \in T$ and $x \in M$, $(sx)t = s(xt)$.

If M is a right S-system and N a left S-system, let τ be the equivalence relation on $M \times N$ generated by $\{((xs, y), (x, sy))\colon x \in M,\ s \in S,\ y \in N\}$. We denote $(M \times N)/\tau$ by $M \otimes_S N$, the *tensor* product over S of the two S-systems. The equivalence class $(x, y)\tau$ will be denoted by $x \otimes y$. Note that $xs \otimes y = x \otimes sy$ $(x \in M,\ s \in S,\ y \in N)$.

Observe that if M is a (T, S)-bisystem and N is an (S, U)-bisystem, then $M \otimes_S N$ becomes a (T, U)-bisystem if we define $t(x \otimes y) = tx \otimes y$, $(x \otimes y)u = x \otimes yu$ for $t \in T$, $u \in U$, $x \otimes y \in M \otimes_S N$.

If P and Q are right S-systems, we say that a map $\alpha : P \to Q$ is a *right S-system morphism* if for every x in P and s in S, $(xs)\alpha = (x\alpha)s$. Similar definitions apply to left S-systems and (S,T)-bisystems.

Next, suppose that U is a subsemigroup of a semigroup S and let $S^{(1)}$ be the semigroup obtained from S by adjoining an identity element 1 whether it has one or not, and let $U^{(1)} = U \cup \{1\}$; then $U^{(1)}$ is a subsemigroup of $S^{(1)}$. Clearly we may form $A = S^{(1)} \otimes_{U^{(1)}} S^{(1)}$.

THEOREM 1.2.7[38, Theorem 2.5 Chapter VII]: If U is a subsemigroup of a semigroup S and if $d \in S$, then $d \in \mathrm{Dom}(U,S)$ if and only if $d \otimes 1 = 1 \otimes d$ in $A = S^{(1)} \otimes_{U^{(1)}} S^{(1)}$.

PROOF: Suppose that $d \in S$ and $d \otimes 1 = 1 \otimes d$ in A. The tensor product A is $(S^{(1)} \otimes S^{(1)})/\tau$, where τ is the equivalence relation on $S^{(1)} \times S^{(1)}$ generated by

$$T = \{((xu,y),\ (x,uy)) : \ x,y \in S^{(1)},\ \ u \in U^{(1)}\}.$$

Let R be a semigroup and let $\beta : S \to R$, $\gamma : S \to R$ be morphisms coinciding on U. We can regard β and γ as morphisms from $S^{(1)}$ into $R^{(1)}$ coinciding on $U^{(1)}$ by defining $1\beta = 1\gamma = 1$. Define $\psi : S^{(1)} \times S^{(1)} \to R^{(1)}$

$$(x,y)\psi = (x\beta)(y\gamma)\ (x,y) \in S^{(1)} \times S^{(1)}$$

It is easily checked that $T \subset \psi \circ \psi^{-1}$, since $\psi \circ \psi^{-1}$ is an equivalence relation. Hence the map $\chi : A \to R^{(1)}$ defined by $(x \otimes y)\chi = (x\beta)(y\gamma)$ $(x \otimes y \in A)$ is indeed well-defined. But now $(d \otimes 1)\chi = (1 \otimes d)\chi$; that is $d\beta = d\gamma$ and so $d \in \mathrm{Dom}(U,S)$.

To prove the converse we regard the tensor product A as an $(S^{(1)}, S^{(1)})$-bisystem by defining

$$s(x \otimes y) = sx \otimes y, \quad (x \otimes y)s = x \otimes ys \quad (s,x,y \in S^{(1)}).$$

Let $(Z(A),+)$ be the free abelian group on A. The abelian group $Z(A)$ inherits an $(S^{(1)}, S^{(1)})$-bisystem structure from A if we define

$$s(\Sigma z_i a_i) = \Sigma z_i(Sa_i), \quad (\Sigma z_i a_i)s = \Sigma z_i(a_i s)$$

for all $s \in S^{(1)}$ and $\Sigma z_i a_i \in Z(A)$. Observe that for any $x,y \in Z(A)$ and $s \in S^{(1)}$ we have

$$s(x+y) = sx + sy, \quad (s+y)s = xs + ys. \tag{3}$$

Next, we define a binary operation on $S^{(1)} \times Z(A)$ by

$$(p,x)(q,y) \;=\; (pq,\; py+xq). \tag{4}$$

Using the statements labelled (3) and (4), one verifies that this operation makes $S^{(1)} \times Z(A)$ a semigroup with identity $(1,0)$.

We now consider two homomorphisms β and γ from $S^{(1)}$ into $S^{(1)} \times Z(A)$ and show that $\beta|U = \gamma|U$. We define β by $s\beta = (s,0)$ $(s \in S^{(1)})$. Clearly β is a morphism. We define γ by $s\gamma = (s,\, s(1\otimes 1) - (1\otimes 1)s)$ $(s \in S^{(1)})$. To show that γ is a morphism, we denote $1 \otimes 1$ by a and, using the statements (3) and (4) we verify that $(s, sa - as)(t, ta - at) = (st, s(ta - at) + (sa - as)t) = (st, (st)a - a(st))$.

If $u \in U^{(1)}$, then

$$u(1\otimes 1) \;=\; u\otimes 1 \;=\; 1u\otimes 1 \;=\; 1\otimes u1 \;=\; (1\otimes 1)u$$

and so $u\beta \;=\; u\gamma$. Removing identities gives two morphisms β and γ from S into $S^{(1)} \otimes Z(A)$ such that $u\beta = u\gamma$ for all u in U. If $d \in \mathrm{Dom}(U,S)$ we must therefore have that $d\beta = d\gamma$, that is $d \otimes 1 = 1 \otimes d$ as required.

We may now complete the proof of the zigzag theorem. Take any $d \in \mathrm{Dom}(U,S)$. By Theorem 1.2.7 we have that $d\otimes 1 = 1\otimes d$ in the tensor product $A = S^{(1)} \otimes_{U^{(1)}} S^{(1)}$. Hence the pair $(1,d)$ and $(d,1)$ are connected by a finite sequence of steps of the form

$$(xu,y) \to (x,uy) \tag{5}$$

or of the form

$$(x,uy) \to (xu,y). \tag{6}$$

If we have two successive steps

$$(xu,y) \to (x,uy) \;=\; (zv,uy) \to (z,vuy)$$

of the first type we may achieve the same effect with a single step of this type:

$$(xu,y) \;=\; (zvu,y) \to (z,vuy).$$

A similar remark applies to the other case. Consequently we may assume that steps of the two types occur alternately in the sequence connecting $(1,d)$ to $(d,1)$.

The first and last steps must have the form

$$(1, d) = (1, uy) \to (u, y) \text{ and } (x, u) \to (xu, 1) = (d, 1)$$

respectively.

Hence the statement that $d \otimes 1 = 1 \otimes d$ is equivalent to the statement that $(1, d)$ is connected to $(d, 1)$ by a sequence of steps as follows:

$$\begin{aligned}
(1, d) &= (1, u_0y_1) \to (u_0, y_1) \\
&= (x_1u_1, y_1) \to (x_1, u_1y_1) \\
&= (x_1, u_2, y_2) \to (x_1u_2, y_2) \\
&= \ldots \\
&= (x_iu_{2i-1}, y_i) \to (x_i, u_{2i-1}y_i) \\
&= (x_iu_{2i}, y_{i+1}) \to (x_i, u_{2i}y_{i+1}) \\
&= \ldots \\
&= (x_m, u_{2m}) \to (x_mu_{2m}, 1) = (d, 1)
\end{aligned}$$

where $u_0, \ldots, u_{2m} \in U^{(1)}$, $x_1, \ldots, x_m, y_1, \ldots, y_m \in S^{(1)}$, and where $d = u_0y_1$, $u_0 = x_1u_1$, $u_{2i-1}y_i = u_{2i}y_{i+1}$, $x_iu_{2i} = x_{i+1}u_{2i+1}$ $(i = 1, 2, \ldots, m-1)$, $u_{2m-1}y_m = u_{2m}, x_mu_{2m} = d$.

Without loss we may assume that $u_i \in U$, since a transition of type (5) or (6) with $u = 1$ may be deleted. If any $x_i = 1$, let x_k be the last x_i that is equal to 1. Thus we have a subsequence of the sequence above as follows:

$(1, d) \to \ldots \to (1, u_{2k}y_{k+1})$ (but ending in $(1, u_{2m})$ if $k = m$).

Note that if (p, q) and (r, s) are connected by steps of the form (5) and (6) then $pq = rs$. In the present instance this gives $d = U_{2k}y_{2k+1}$ (or $d = u_{2m}$); hence this subsequence merely connects $(1, d)$ to $(1, d)$ and so may be deleted. What remains is a sequence in which no x_i is 1.

A dual argument now ensures that we may construct a sequence from $(1, d)$ to $(d, 1)$ so that no y_i is 1. This completes the proof of the theorem.

REMARK 1: The above zigzag theorem is also valid in the category of all commutative semigroups (Howie and Isbell [40]).

RESULT 1.2.8[46, Result 4]: Let U and S be any semigroups with U a subsemigroup of S and $\mathrm{Dom}(U,S) = S$. Then for any $d \in S\backslash U$ and any positive integer k, there exist $a_1, a_2, \ldots, a_k \in U$ and $d_k \in S\backslash U$ such that $d = a_1 a_2 \ldots a_k d_k$. In particular $d \in S^k$ for each positive integer k.

PROOF: Since $d \in S\backslash U$ and $\mathrm{Dom}(U,S) = S$, by Result 1.2.5, there exist $a_1 \in U$, $d_1 \in S\backslash U$ such that $d = a_1 d_1$. Applying again Result 1.2.5, but this time to d_1, we get $d = a_1 a_2 d_2$ for some $a_2 \in U$ and $d_2 \in S\backslash U$. Continuing this process gives us the required result.

RESULT 1.2.9[46, Result 3]: Let U and S be any semigroups with U a subsemigroup of S. Take any $d \in S\backslash U$ such that $d \in \mathrm{Dom}(U,S)$. Let (1.2) be a zigzag of shortest possible length m over U with value d. Then $t_j, y_j \in S\backslash U$ for $j = 1, 2, \ldots, m$.

PROOF: Since $d \in S\backslash U$, obviously $t_1 \in S\backslash U$. So let us suppose to the contrary that $t_k \in U$ for some $k \in (2, 3, \ldots, m)$.

Then

$$\begin{aligned}
d &= a_0 t_1, \quad a_0 = y_1 a_1 \\
y_i a_{2i} &= y_{i+1} a_{2i+1}, \quad a_{2i-1} t_i = a_{2i} t_{i+1} \quad (i = 1, 2, \ldots, k-2) \\
a_{2k-3} t_{k-1} &= a_{2k-2} t_k \in U, \quad y_{k-1}(a_{2k-2} t_k) = d
\end{aligned}$$

is a zigzag of length $k - 1 < m$, a contradiction as required.

In the following result as another application of the zigzag theorem, we show that Dominion of a commutative semigroup is commutative, the result due to Isbell [42]. An elegant proof of the corresponding fact for the rings using the zigzag theorem for rings can be found in Bulazewska and Krempa [5].

RESULT 1.2.10[42, Corollary 2.5]: If U is a commutative subsemigroup of any semigroup S, then $\mathrm{Dom}(U,S)$ is also commutative.

PROOF: Let $d \in \mathrm{Dom}(U,S)$ and let z be a zigzag with value d. Let $u \in U$. Then $ud = uu_0 y_1 = u_0 u y_1 = x_1 u_1 u y_1 = x_1 u u_1 y_1$

$$x_1 u u_2 y_2 = \ldots = x_m u_{2m-1} u y_m = x_m u u_{2m-1} y_m = x_m u u_{2m} = du.$$

Hence every element of U commutes with every element of $\mathrm{Dom}(U, S)$. It remains to show that if d and d' are members of $\mathrm{Dom}(U, S)$ then $dd' = d'd$. Once again take the zigzag z for d. Since d' commutes with all members of the spine of z, the manipulation above can now be repeated to yield the conclusion that $\mathrm{Dom}(U, S)$ is commutative.

A morphism $\alpha : A \to B$ from A to B in the category C of all semigroups is called an epimorphism (epi for short) if for all $c \in C$ and for all morphism $\beta, \gamma : B \to C$, $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$. The following facts can be easily proved. A morphism $\alpha : S \to T$ is epi if and only if the inclusion mapping $i : S\alpha \to T$ is epi, and an inclusion mapping $i : U \to S$ is epi if and only if $\mathrm{Dom}(U, S) = S$.

In such a case we say that U is epimorphically embedded in S or S is an epimorphic extension of U. It is easy to see that every onto morphism is an epimorphism, but the converse is not true in general in the category of all semigroups.

An example of a semigroup epimorphism (epi for short) which is not onto appears in Drbohlav [11]. Take the embedding i of the real interval $(0, 1]$ into $(0, \infty)$, where both are considered as multiplicative semigroups. To see that $i : (0, 1] \to (0, \infty)$ is epi, take any pair of homomorphisms α, β from $(0, \infty)$ such that $i\alpha = i\beta$; that is α and β agree on $(0, 1]$. We shall show that for any $x > 1$, $x\alpha = x\beta$ by employing a "zigzag" argument. Let $x > 1$. Then

$$\begin{aligned}
[(x)\alpha \cdot (1/x)\alpha](x)\beta &= (1)\alpha \cdot (x)\beta \\
&= (1)\beta \cdot (x)\beta \\
&= (x)\beta.
\end{aligned}$$

Equally through, since $1/x < 1$,

$$\begin{aligned}
[(x)\alpha \cdot (1/x)\alpha](x)\beta &= (x)\alpha[(1/x)\alpha \cdot (x)\beta] \\
&= (x)\alpha[(1/x)\beta \cdot (x)\beta] \\
&= (x)\alpha \cdot (1)\beta \\
&= (x)\alpha \cdot (1)\alpha \\
&= (x)\alpha.
\end{aligned}$$

Therefore $\alpha = \beta$ and i is epi.

The embedding of an infinite cyclic semigroup into an infinite cyclic group, and the embedding (under multiplication) of the natural numbers into the positive rational numbers are other examples of epimorphisms which are not onto. More generally Hall [39], has noted that if U is a full subsemigroup (containing all idempotents of S) of an inverse semigroup S, which generates S as an inverse semigroup, then the embedding of U in S is an epimorphism. This observation unifies all the above examples and shows that they are essentially the one and same. The paucity of examples other than of this kind was one of the main difficulties encountered in this work.

Whether or not epis are onto depends on the category under consideration. They are onto, in the categories of sets, abelian groups, groups and regular rings for instance. The proofs are respectively trivial, easy and hard (Burgess [4]). In other categories the converse is false, although a simple alternative description is sometime available, notably for the categories of fields and torsion free abelian groups (Burgess [4]).

In general epimorphisms are not onto in the categories of rings and the semigroups: here epimorphisms can be characterized in terms of so called "zigzag", a special sequence of factorizations of the elements in the epimorphic image. In this dissertation we shall work in the category of semigroups. For further references on the relevant topics, one could see [1-3, 8, 9, 12-20, 22-27, 29, 32, 37, 41, 50-72].

CHAPTER 2

NECESSARY CONDITION FOR A SEMIGROUP VARIETY TO BE SATURATED AND EPIMOPHICALLY CLOSED

2.1. INTRODUCTION

A semigroup U is said to be absolutely closed if $\mathrm{Dom}(U,S) = U$ and saturated if $\mathrm{Dom}(U,S) \neq S$ for every properly containing semigroup S. A variety of semigroups will be called absolutely closed if every member of V is absolutely closed and saturated if every member of V is saturated. Further a variety V of semigroups is called epimorphically closed or closed under epis if whenever Dom $(U,S) = S$ for any semigroup $U \in V$ implies $S \in V$. Clearly all absolutely closed and saturated varieties are epimorphically closed. Absolutely closed and saturated varieties have been studied in [33], [43] and [45]

We, then, by constructing examples of semigroups, give necessary conditions for any semigroup variety to be saturated and to be epimorphically closed. The results of this chapter are essentially due to P.M Higgins [33].

2.2. NECESSARY CONDITION FOR A SEMIGROUP VARIETY TO BE SATURATED

In this section we give a necessary condition for a semigroup to be saturated, by constructing an example of a semigroup.

EXAMPLE 2.2.1: Let V be the variety of commutative semigroups defined by the identity $x^2y = x^2$ (note that this identity is equivalent to $x^2 = 0$). We construct a semigroup S in V which has a proper subsemigroup U such that $\mathrm{Dom}(U,S) = S$. By, Howie and Isbell ([40], Theorem 2.2), any such semigroup U can not satisfy the minimum condition on principle ideals and so in particular can not be finite.

Let F be the free commutative semigroup in the variety V on the countably infinite set $X = \{x_1, x_2, \ldots\ldots\}$ of generators. It may be thought of F as consisting of finite subsets of X together with the additional symbol 0 with multiplication defined by

$$AB = \begin{cases} A \cup B & ; if A \cap B = \phi \\ 0 & ; otherwise \end{cases}$$

Let $S = F/\rho$, where ρ is the congruence generated by the relation ρ_0 on F which consists of the pairs

$(\{x_{3n-2}\}, \{x_{6n-2}, x_{6n-1}, x_{6n}\}), (\{x_{3n}\}, \{x_{6n+1}, x_{6n+2}, x_{6n+3}\})$

for all $n = 1, 2, 3, \ldots\ldots$. Next let U be the subsemigroup of S generated by

$$\bigcup_{n=1}^{\infty} \{\{x_{3n-2}, x_{3n-1}\}, \{x_{3n-1}\}, \{x_{3n-1}, x_{3n}\}\}$$

.

Then it is shown that $\mathrm{Dom}_C(U, S) = S$, where C is the variety of all commutative semigroups.

To verify this it suffices to show that $\{x_i\} \in Dom_C(U, S)$ for all generators $\{x_i\}$ of S. The generators $\{x_{3n-1}\}$, $n = 1, 2, 3, \ldots$ are members of U and so are certainly in its dominion. A generator indexed by $3n$, for $n = 1, 2, 3, \ldots$, has a zigzag in S over U :

$$\begin{aligned} \{x_{3n}\} &= (\{x_{6n+1}, x_{6n+2}\})\{x_{6n+3}\} \\ &= \{x_{6n+1}\}(\{x_{6n+2}\})\{x_{6n+3}\} \\ &= \{x_{6n+1}\}(\{x_{6n+2}, x_{6n+3}\}) \end{aligned}$$

where the bracketed terms form the spine of the zigzag. A similar zigzag exists for any generator indexed by some x_{3n-2}.

The author (P.M. Higgins) establishes that U is not saturated by proving that $U \neq S$. This is done by showing that $\{x_1\} \notin U$. This is proved by analysing the form of members of $F\rho$-related to $\{x_1\}$ and, then, noting that no product of elements from U has this form.

An elementary ρ_o transition of the form $A\{x_{3n-2}\}\ B \rightarrow A\{x_{6n-2}, x_{6n-1}, x_{6n}\}B$ or $A\ \{x_{3n}\}B \rightarrow A\{x_{6n+1}, x_{6n+2}, x_{6n+3}\}B$ where $A,\ B \in F^1$ is known as an *upward transition* based on $\{x_{3n-2}\}$ or $\{x_{3n}\}$ respectively while their reversals are called *downward transitions* based on $\{x_{3n-2}\}$ or $\{x_{3n}\}$ as the case may be.

LEMMA 2.2.2: Let $Y\rho\{x_1\}$, $Y \neq \{x_1\}$ and let $I : \{x_1\} \rightarrow \ldots\ldots \rightarrow Y$ be a sequence

of elementary transitions from $\{x_1\}$ to Y. Then there exists a sequence
J : $\{x_1\} \to ...... \to Y$ consisting of upward transitions.

PROOF: Suppose that in I, there is an upward transition followed by a downward transition. There are four cases :

(i) $w_1x_{3n-2}w_2 \to w_1x_{6n-2}x_{6n-1}x_{6n}w_2 = w_3x_{6m-2}x_{6m-1}x_{6m}w_4 \to w_3x_{3m-2}w_4$;

(ii) $w_1x_{3n-2}w_2 \to w_1x_{6n-2}x_{6n-1}x_{6n}w_2 = w_3x_{6m+1}x_{6m+2}x_{6m+3}w_4 \to w_3x_{3m}w_4$;

(iii) $w_1x_{3n}w_2 \to w_1x_{6n+1}x_{6n+2}x_{6n+3}w_2 = w_3x_{6m+1}x_{6m+2}x_{6m+3}w_4 \to w_3x_{3m}w_4$;

(iv) $w_1x_{3n}w_2 \to w_1x_{6n+1}x_{6n+2}x_{6n+3}w_2 = w_3x_{6m-2}x_{6m-1}x_{6m}w_4 \to w_3x_{3m-2}w_4$.

In all cases the two transitions either cancel (e.g. in case (i) if $m = n$) or can be performed in the opposite order without changing the net effect. Applying this argument repeatedly to I, we obtain a sequence J in which either there are no downward transitions or all downward transitions are at the beginning. Since I begins with $\{x_1\}$, this latter alternative is impossible.
Hence the lemma is proved.

LEMMA 2.2.3: Let $Y\rho\{x_1\}$, $Y \neq \{x_1\}$ and let :$\{x_1\} \to ... \to Y$ be a sequence consisting of upward transitions from $\{x_1\}$ to Y. Then the following conditions are satisfied:

(i) If one of x_{6n-2}, x_{6n-1}, $x_{6n}[x_{6n+1}$, x_{6n+2}, $x_{6n+3}]$ is a member of Y, then I contains a transition based on $x_{3n-2}[x_{3n}]$ and $x_{3n-2} \notin Y[x_{3n} \notin Y]$,

(ii) An upward transition based on $x_{3n-2}[x_{3n}]$ occurs at most once in I for each n=1,2,3,...,

(iii) $Y \neq 0$.

PROOF: We proceed by induction on $|I|$, the number of transitions in I. If $|I| = 1$, then I is given by $I : \{x_1\} \to \{x_4,\ x_5,\ x_6\}$ and the above three conditions are then evidently satisfied. Next suppose that $|I| > 1$ and I is given by the sequence $I : \{x_1\} \to ... \to Y' \to Y$ and call the sequence of upward transitions $\{x_1\} \to ... \to Y'$ by the name J. We take as our inductive assumption that the three conditions hold for Y' and J.

We consider the transition $Y' \to Y$ which we suppose is based on some x_{3n} (the x_{3n-2} case being similar), so $Y' \to Y$ has the form $p\{x_{3n}\}q \to p\{x_{6n+1}, x_{6n+2}, x_{6n+3}\}q$ for some $n \geq 1$ $p, q \in F^1$. If condition (i) were voilated, then this would imply that either $x_{3n} \in Y$ or that Y contains some generator x_i introduced by transitions based on x_{6n+1} or x_{6n+3} (note there are no transition based on x_{6n+2}). This first case does not arise as $Y' \to Y$ is based on x_{3n}. If the second case arises, this would imply that a transition based on x_{6n+1} or x_{6n+3} preceded $Y' \to Y$. This in turn implies that an upward transition based on x_{3n} preceded $Y' \to Y$ which like wise implies that two upward transitions introducing x_{3n} occurred in J, which are necessarily both based on the same generator, contradicting condition (ii)for J. Therefore condition (i) is satisfied by Y. Similarly to show (ii), we suppose to the contrary that a transition based on x_{3n} has occurred previously in J. This implies that J contains at least two upward transitions which introduce x_{3n}, again contradicting condition (ii) for J. It is immediate that $Y \neq 0$ as from condition (i) it follows that each of the letters x_{6n+1}, x_{6n+2}, x_{6n+3} are not members of Y'. This completes the proof.

we can now show that $\{x_1\} \notin U$. Let $A \in F$ be a product of members of the set $\bigcup_{n=1}^{\infty} \{x_{3n-2}\ , x_{3n-1}\}\ , \{x_{3n-1}\}\ , \{x_{3n-1}\ , x_{3n-2}\}$. If $A = 0$, then $(A, \{x_1\}) \notin \rho$ by Lemma 2.4. If $A \neq 0$, then for all $n = 1, 2, \cdots$ at least one of the sets $\{x_{3n-2}\ , x_{3n-1}\}$ or $\{x_{3n-1}\ , x_{3n}\}$ is not contained in A. However, from Lemma 2.3, it follows that if $Y\rho\{x_1\}$, then $\{x_{3n-2}\ , x_{3n-1}\ , x_{3n}\} \subseteq Y$ for some $n = 1, 2, \cdots$, (consider the final transition in any sequence of upward transitions $I : \{x_1\} \longrightarrow \cdots \longrightarrow Y$.) Therefore $(A, \{x_1\}) \notin \rho$. It follows that $\{x_1\} \notin U$.

We now find the variety of commutative semigroups generated by U. By a "non-trivial" identity we mean an identity not implied by the conjuction of the associative and commutative laws.

LEMMA 2.2.4: The semigroup U satisfies a non-trivial identity k if and only if each side of k has a variable which appears at least twice.

PROOF: If k is an identity in which both sides contain a repeated variable, then substituting any member of U for the variable of k which occurs at least twice gives the statement $0 = 0$. Hence U satisfies k.

Conversely, observe that the subsemigroup of U generated by $\{\{x_2\}, \{x_5\}, .., \{x_{3i-1}\}, \cdots\}$ is a copy of F because no ρ_0- transition is possible from any non-zero product of these generators. Now suppose from any non-zero product of these generators. Now suppose k is any identity in m distinct variables for which exactly one side contains no repeated variable. Then substituting each of the variables of k with the letters $x_2,\ x_5,\ \cdots x_{3m-1}$ we derive the false statement that the some subset of $\{x_2,\ x_5,\ \cdots x_{3n-1}\} \ = \ 0$. Hence U does not satisfy k. Finally suppose k is a non trivial identity for which each side has no repeated variable. Then k has the form

$$x_1x_2\cdots x_ny_1y_2\cdots y_m \ = \ x_1x_2\cdots x_nz_1z_2\cdots z_k$$

where $m, n \geq \ 0,\ k \geq 1$. Then k implies the identity k'

$$x_1x_2\cdots x_ny_1y_2\cdots y_m \ = \ x_1x_2\cdots x_nz_1z_2\cdots z_k^2$$

and by above, U does not satisfyk' and so U does not satisfy k.
Now the following theorem gives a necessary condition for a semigroup variety to be saturated.

THEOREM 2.2.5: Let V be a variety not equal to the variety of all semigroups. The variety V is saturated only if (S) every presentation of V contains an identity, not a permutation identity, for which at least one side contains no repeated variable. Condition (S) is equivalent to the condition that V admits a homotypical identity of the form

$$x_1x_2\cdots x_n \ = \ f(x_1, x_2, \cdots, x_n) \ \ with \quad |x_I|_f \ > \ 1$$

or some x_i.

PROOF: The semigroup U of Example 2.2.1 satisfies every identity for which both sides contain repeated variables. If V is saturated, then $U \notin V$, and so each presentation of V must include an identity, not a permutation identity, for which at least one side contains no repeated variable, that is V must satisfies condition (S).
To show the equivalence of condition (S) and the given condition we take V to be a variety admitting an identity $\phi\ :\ x_1x_2\cdots x_n \ = \ f(x_1, x_2, \cdots, x_m)$, not a permutation identity, for which at least one side contains no repeated variable. Replace each variable on the right which does not occur on the left by x_1, to get the identity $\phi' : x_1x_2\cdots x_n \ = \ f'(x_1, x_2, \cdots, x_n)$. Next suppose x_i is a variable which occurs on the left of the ϕ', but not on the right. It follows that the value of $x_1x_2\cdots x_n$ is independent of the value assigned to x_i,

so we get, on replacing x_i by $x_i x_{i+1} \cdots x_n x_i$ in ϕ'

$$\begin{aligned} x_1 x_2 \cdots x_i \cdots x_n &= x_1 x_2 \cdots (x_i x_{i+1} \cdots x_n x_i) x_{i+1} \cdots x_n \\ &= f'(x_1, x_2, \cdots, x_n) x_i x_{i+1} \cdots x_n. \end{aligned}$$

Therefore ϕ implies the identity $x_1 x_2 \cdots x_n = f'(x_1, x_2, \cdots, x_n) x_i x_{i+1} \cdots x_n$, and repeating this procedure for each such x_i eventually yields a homotopical identity of the required type.

Conversely take any variety V admitting a homotypical identity of the form $x_1 x_2 \cdots x_n = f(x_1, x_2, \cdots, x_n)$ *with* $|x_i|_f > 1$ for some i. By Lemma 2.2.4 we have that $U \notin V$, whence it follows again from Lemma 2.2.5, that every presentation for V must contain an identity of the required type.

2.3. NECESSARY CONDITION FOR EPIMORPHICALLY CLOSED VARIETIES

In this section we find a necessary condition for a semigroup variety to be epimoriphically closed by constructing an example of a semigroup satisfying all identities of which both sides contain repeated variables and, then, showing that its epimorphic extension does not satisfy them.

Let u be any word. The content of u is the (necessarily finite) set of all distinct variables appearing in u, and will be denoted by $C(u)$. Further for any variable x of u, $|x|_u$ will denote the number of occurrences of the variable x in the word u.

An identity $u(x_1, x_2, ..., x_n) = v(x_1, x_2, ..., x_n)$ is called homotypical if $C(u) = C(v)$ and heterotypical otherwise.

Clearly, saturated and absolutely closed varieties are epimorphically closed. Saturated and absolutely closed varieties have been studied in [21], [34], [35], [36], [39], [46], [47] and [48].

We may state the main result of this section.

THEOREM 2.3.1: Let V be a variety of semigroups not equal to the variety of all semigroups. Then

(i) The variety V is epimorphically closed only if

(E) Every presentation of V contains a nontrivial identity for which one side contains no repeated variable.

Condition (E) is equivalent to the condition that V admits a nontrivial homotypical identity of the form

$$x_1x_2....x_n = f(x_1x_2....x_n).$$

PROOF: (i) To prove this we construct an example of a semigroup S which satisfies no non-trivial identity which is dominated by a subsemigroup U, which satisfies an identity ϕ if and only if both sides of ϕ contain a repeated variable.

EXAMPLE 2.3.2[34]: We construct a semigroup S, which satisfies no nontrivial identity, and which is dominated by a subsemigroup U, which satisfies an identity ϕ if and only if both sides of ϕ contains a repeated variable.

Let F be the free semigroup on the countably infinite set of generators $\{x_1, x_2, ..., a_1, a_2, ..., y_1, y_2, ...\}$. Let A be the subsemigroup of F generated by $\{a_1, a_2.....\}$. Let ρ_o be the relation on F consisting of the pairs (wv,w) and (vw,w) for all words v of A, and all words w of A containing a repeated letter, together with pairs defined by the zigzags:

$x_n = a_{6n-2}y_{2n} = x_{2n}a_{6n-1}y_{2n+1} = x_{2n}a_{6n}$ for all n =1,2,.... and

$y_n = a_{6n+1}y_{2n+1} = x_{2n+1}a_{6n+2}y_{2n+1} = x_{2n+1}a_{6n+3}$ for all n =1,2,.... that is

$(x_n, a_{6n-2}y_{2n}), (a_{6n-2}, x_{2n}a_{6n-1}), (a_{6n-1}y_{2n}, a_{6n})$ for all n = 0,1,2...

$(y_n, a_{6n+1}y_{2n+1}), (a_{6n+1}, x_{2n+1}a_{6n+2}), (a_{6n+2}y_{2n+1}, a_{6n+3})$ for all n = 0,1,2....

We note that y_o is a symbol denoting a_1y_1 and is not a generator of F. Let ρ be the congruence generated by ρ_o and put $S = F/\rho$ and $U = A\rho$. By construction $Dom(U, S) = S$, and note that all words of A with a repeated letter are in the same ρ-class which forms the O of U. We next show that $U \neq S$ by showing that $Y_o\rho \notin U$.

First we introduce some convenient definitions. For an arbitrary elementary ρ_o - transition $puq \to pvq$ with $p, q \in F^1$, we call u the base and v the replacement of the transition. Elementary transitions of the type $pwvq \to pwq$ or $pvwq \to pwq$, where $p, q \in F^1$ and their reversals , will be known as zero transitions. By a forward transition, we shall mean one of the type

$pa_{6n-2}q \to px_{2n}a_{6n-1}q, pa_{6n-1}y_{2n}q \to pa_{6n}, pa_{6n+1}q \to px_{2n+1}a_{6n+2}q or pa_{6n+2}y_{2n+1}q \to pa_{6n+3}q$ while the corresponding reversals shall be known as backward transitions. Transitions of the type $px_nq \to pa_{6n-2}y_{2n}q$ and $py_nq \to pa_{6n+1}y_{2n+1}q$ shall be called upward transitions and their reversals shall be called downward transitions. Collectively, upward and forward transitions shall be called positive transitions while the backward and downward transitions shall be called negative transitions. A sequence of elementary transitions I shall be called positive if it consists entirely of positive transitions . A set of the form $\{a_{3n-2}, a_{3n-1}, a_{3n-2},\}$, n = 1,2... shall be called companion set and each member of the set is a companion of the other two. The companion sets correspond to the spines of the above zigzags. Suppose V is an epimorphically closed variety which has a presentation I, and further suppose that for each nontrivial member of I, both sides contain a repeated variable. Then by example $U \in V$, whence $S \in V$, whence V is the variety of all semigroups. We may then establish the equivalence of condition (E).

LEMMA 2.3.3: Suppose $w\rho a_1y_1$ and $I : a_1y_1 \to .....w^1 \to w$ be a shortest possible sequence of elementary ρ_0- transitions from a_1y_1 to w. Then the following conditions are satisfied:

(i) w contains no repeated letter;

(ii) w does not contain any two members from any one companion set;

(iii) I is positive ;

(iv) no two transitions of I have the same base and any base of a transition in I does not occur in w;

(v) there is a factorization $w = w_1w_2w_3$ of w in F^1 such that

(a) $w_2 = x_m$ or $w_2 = a_{3m-2}y_m$, for some $m \geq 1$,

(b) if $w_2 = x_m$ for $m \geq 1$, then $w_3 \neq 1$,

(c) w_3 is a product of words a_{3n}, n = 1,2,3......, and $a_{3n-1}y_n$, n = 1,2,3....;

(vi) there is a factorizations $w = v_1v_2v_3$ of w in F^1 such that

(a) $v_2 = y_m$ or $v_2 = x_ma_{3m}$, for some $m \geq 1$,

(b) if $v_2 = y_m$ for some $m \geq 1$, then $v_1 \neq 1$,

(c) v_1 is a product of words a_{3n-2}, n = 1,2,3.... and x_na_{3n-1}, n = 1,2,3,.....

REMARK 1: It follows at once from either conditions (v) or (vi) that $C(w) \not\subseteq A$ *and so* $y_0 \notin U$ as required.

PROOF OF THE LEMMA: We proceed by induction on $|I|$, the number of transitions in I. If $|I| = 0$, then conditions (i) to (vi) are evidently satisfied. Consider an arbitrary shortest sequence I and suppose the lemma holds for the initial subsequence $J : a_1y_1 \to w'$ of I with w'_1, w'_2, w'_3, and v'_1, v'_2, v'_3, being subwords of w' satisfying conditions (v) and (vi) respectively.

We shall consider the transition $w' \to w$ which is either (1) a zero transition, (2) an upward transition based on some x_n or y_n, (3a) a forward transition based on a_{3n-2} for some $n \geq 1$ or (3b) a forward transition based on $a_{3n-1}y_n$ for some $n \geq 1$ or (4) a negative transition.
We shall show that cases (1) and (4) do not arise while in cases (2),(3a) and (3b), the conditions (i) to (vi) of the lemma continue to hold.
By conditions (i) applied to w', the transition $w' \to w$ can not be a zero transition, thereby eliminating case(1).

Next consider case (2) and suppose $w' \to w$ has the form $px_nq \to pa_{6n-2}y_{2n}q$(the case where $w' \to w$ is based on some y_n is similar).

By condition (vi), no transition based on x_n has occurred in$J$ and since x_n is the base of the unique positive transition which introduces either of the letters a_{6n-2} or y_{2n} it follows that w has no repeated letters, that is, w satisfies condition (i). Since the unique positive transition which introduces a_{6n-1} is based on a_{6n-2}, it follows that $a_{6n-1}, a_{6n} \notin C(w)$ and so condition (ii) is satisfied by w, and of course that condition (iii) is satisfied is clear, while condition (iv) follows from the facts that the letters $a_{6n-1}y_{2n}$ have not appeared in J and w' has no repeated letters, so that $x_n \notin C(w)$. For condition (v) we note that if w'_2 is in q, then we may take $w_i = w'_i$, i = 2,3; otherwise $w'_1 = p$, $w'_2 = x_n$, and $w'_3 = q$ whence we can take $w_1 = w'_1$, $w_2 = a_{6n-2}y_{2n}$ and $w_3 = w'_3$. To prove condition(vi), we note that v'_2 can serve as v_2 if v'_2 occurs in p, otherwise v'_2 occurs in x_nq whence we may put $v_1 = pa_{6n-2}q$, $v_2 = y_{2n}$ and $v_3 = q$.

Next we consider case (3a) where $w' \to w$ has the form $pa_{3n-2}q \to px_na_{3n-1}q$ for some $n \geq 1$. Since a_{3n-2} and a_{3n-1} are companions, it follows that a_{3n-1} is not a repeated letter by condition (ii). If x_n were repeated, then since J has no negative transitions, this would imply that a forward transition based on a_{3n-2} occurred in J, which contradicts condition (iv), as a_{3n-2} appears in w'. Hence w satisfies condition (i). Conditions (ii) and (iii) are clearly satisfied. The condition (ii) also follows from the fact that a_{3n-2} and a_{3n-1} are companions. Condition (iv) applied to w' shows that no transition based on a_{3n-2} occurred in J, which implies condition (iv) holds for I. To show condition (v) holds, we note that if w'_2 occurs in q, then w'_2 can serve as w_2 in w while w'_2 can not occur in p. The remaining case is where $pa_{3n-2}q = pa_{3n-2}y_nq'$ where $y_nq' = q$ and $w'_2 = a_{3n-2}y_n$. We take $w_1 = p$, $w_2 = x_n$ and $w_3 = a_{3n-1}q$. As for condition (vi), v'_2 must occur either in p or q and in either case we may take $v_2 = v'_2$. In case (3b), $w' \to w$ has the form $pa_{3n-1}y_nq \to pa_{3n}q$. conditions (i)to (iv) then follow in the same way as for case (3a), while for condition (v), w'_2 occurs in either p or q and can serve as w_2 in w. As for condition (vi), we note that v'_2 occurs either in p, in which case it may serve as v_2 or $pa_{3n-1}y_nq = p'x_na_{3n-1}y_nq$ where $p = p'x_n$ with $v'_1 = pa_{3n-1}$, $v'_2 = y_n$ and $v'_3 = q$ whence we may take $v_1 = p'$, $v_2 = x_na_{3n}$ and $v_3 = q$

Finally, we consider case (4) where we suppose $w' \to w$ is a negative transition of the form $puq \to pvq$. In this case a positive transition of the form $p'uq' \to p'vq'$ for some $p', q' \in F^1$ has occurred in J, and by condition (iv) this is the unique transition of J based on v. Hence no word of J preceding $p'uq'$ contains u. Observe that J contains no transition based on a subword of u, as this would contradict condition (iv) since u occurs in w'. This allows us to construct a new sequence, $I' : a_1y_1 \to .... \to p'vq' \to .... \to w$, whose transitions are based on the corresponding transitions of J, but with the transition $p'vq' \to p'uq'$ deleted. In detail,I' is identical to J up to and including the appearance of the word $p'vq'$, and the words in I' appearing after $p'uq'$ correspond to the words of J appearing after $p'uq'$, except that in the words of I' the subword v appears instead of u. However $|I'| = |I| - 2$, contradicting our choice of I and so we conclude case (4) does not arise, thus completing the proof.

LEMMA 2.3.4: The semigroup U satisfies an identity ϕ if and only if both sides of ϕ contain a repeated variable.

PROOF: An identity ϕ in which both sides contain a repeated variable is satisfied by U, as both sides become 0 upon substitution of the variable of ϕ with any members of U.

Conversely, the relative free semigroup of U, generated by $\{a_2, a_5, \ldots\ldots, a_{3n-1}\}$ is a relatively free semigroup on countably infinitely many generators satisfying all identities for which both sides contain a repeated variable, for if w is a product of this set without repeats, there are no nontrivial $\rho_\circ$-transitions from w. This completes the proof.
Since U is properly epimorphically embedded in S, it follows that no variety containing U is saturated. This observation together with Lemma 2.5 implies that any variety V which only admits nontrivial identities for which both sides contain a repeated variable contains U, and so is not saturated. We shall show that any such variety V is not epimorphically closed (unless it is the variety of all semigroups) by showing that S generates the variety of all semigroups. We prove this by showing that the semigroup S^1 of S generated by $y_\circ$ and $a_2y_\circ$ is a free semigroup on two generators, and so contains a free semigroup on countably infinitely many generators ([13], Theorem 1) which satisfies no nontrivial identity.

LEMMA 2.3.5: Let w be a word of F, u be an arbitrary product of a_2y_0 and y_0 which may be write as

$$(a_2y_0)^{m(1)}y_0^{n(1)}\ldots(a_2y_0)^{m(k)}y_0^{n(k)}$$

where $m(1) \geq 0$, $m(i) \geq 1$ for all $1 < i \leq k$,$n(i) \geq 1$ for all $1 \leq i < k$,$n(1) \geq 0$. Then $w\rho u$ if and only if there is a factorization $w = r_1s_1r_2s_2\ldots\ldots r_ks_k$ such that

(a) for all $1 \leq i \leq k$, each r_i admits a factorization

$$r_i = a_2p_{i_1}a_2p_{i_2}\ldots\ldots a_mp_{i_{m(i)}},$$

where each $p_{i_j}\rho y_\circ$ and

(b)for all $1 \leq i \leq k$, each s_i admits a factorization

$$s_i = q_{i_1}q_{i_2}\ldots\ldots q_{i_{n(i)}},$$

where each $q_{i_j}\rho y_\circ$

REMARK 2: The statement of the lemma says that $w\rho u$ if and only if w has the same form as that given for u, with each instance of y_0 replaced by some word ρ-related to y_0.

PROOF: The if part of the statement is immediate. To prove the converse, let$I$: $u \to \ldots\ldots w' \to w$ be a sequence of elementary transitions from u to w, and we assume inductively that w' can be factorised in the manner of the statement of the lemma. we establish the lemma by showing that the base of the transition $w' \to w$ is contained in one of the p^{1s} *or* q^{1s} occuring in this factorization . If this were not the case, the base of $w' \to w$ would be one of the following:

(1) a word of A containing a repeated letter;

(2) a_2t where t is the first letter of some p;

(3) ta_2 where t is the last letter of some p or

(4) t_1t_2 where t_1 is the last letter of some $p_{m(k)}$ or q and t_2 is the first letter of the following q.

In general, if $v\rho y_o$ and v=v'a where a is a word in A, then by 2.4(v) we have $C(a) \subseteq \{a_{3n}, n = 1, 2, ...\}$ and dually , if v = av' where a is a word of A then by 2.4 (vi) we have $C(a) \subseteq \{a_{3n-2}, n = 1, 2, \cdots\}$. These sets are disjoint and a_2 is not a member of either. From this, and the fact that any word ρ related to y_0 has no repeated letter (2.4(i)), it follows that w' contains no word of A with a repeated letter, and hence the transition $w' \to w$ is not a zero transition, and therefore case (1) does not arise. For case (2) to arise we would have $t = y_1$, but no word ρ related to y_0 begins with y_1 by 2.4 (vi) so this is impossible. Similarly, case (3) can not happen, as no word ρ related to y_0 ends with x_1 by 2.4 (v). Lastly, case (4) does not arise as no word ρ related to y_0 begins with an a_{3n-1} by 2.4 (vi), nor ends with an a_{3n-1} or a_{3n-2} by 2.4 (v).

LEMMA 2.3.6: The semigroup S satisfies no nontrivial identity.

PROOF: We show that the subsemigroup S' of S generated by $\{y_0, a_2y_0\}$ is freely generated by this pair. Let $u, v \in S'$ and suppose $u\rho v$ with $u = (a_2y_0)^{m(1)}y_0^{n(1)} \cdots (a_2y_0)^{m(k)}y_0^{n(k)}$, $v = (a_2y_0)^{s(1)}y_0^{t(1)} \cdots (a_2y_0)^{s(r)}y_0^{t(r)}$. By Lemma 2.6, u can be factorized in the form given for v with the y_0's replaced by words ρ related to y_0. However, by Lemma 2.4 (iv), if $p\rho y_0$ and y_0 occurs in p then $p = y_0$, and since $(y_0, a_2) \notin \rho$ each p and q occuring in this factorization of u contains y_0, and so equals y_0, which implies that $u = v$, as required.

CHAPTER 3

EPIS AND PERMUTATION IDENTITIES

INTRODUCTION

In this chapter we extend Result 1.2.5 from commutativity to permutation identities. In [42], Isbell has shown that commutativity is preserved under epis. We prove that all permutation identities are preserved under epis.

A semigroup S satisfying a non-trivial permutation identity will be called a permutative semigroup while a variety V admitting a non-trivial permutation identity is called a permutative variety. An identity $u = v$ is said to be preserved under epis if for all semigroups U and S with U a subsemigroup of S and such that $\mathrm{Dom}(U, S) = S$, U satisfying $u = v$ implies S satisfies $u = v$.

THEOREM 3.1[41, Theorem 2.1]: All permutation identities are preserved under epis.

PROOF: Let

$$x_1 x_2 \ldots x_n \;=\; x_{i_1} x_{i_2} \ldots x_{i_n} \tag{1}$$

be any permutation identity with $n \geq 3$. Without loss we can assume that (1) is nontrivial. Take any semigroup U satisfying (1) and any semigroup S containing U properly and such that $\mathrm{Dom}(U, S) = S$. We shall show that S also satisfies (1).

For $k = 1, 2, \ldots, n$, consider the word $x_{i_1} x_{i_2} \ldots x_{i_k}$ of length k. We shall prove the theorem by induction on the length of these words, assuming that the remaining elements $x_{i_{k+1}}, \ldots, x_{i_n} \in U$.

First for $k = 1$, that is when $x_{i_1} \in S$ and $x_{i_2}, \ldots, x_{i_n} \in U$, we wish to show that equation (1) holds. When $x_{i_1} \in U$, (1) holds trivially; so we assume that $x_{i_1} \in S \backslash U$. By Result 1.2.5, we may let (1.2) be a zigzag of shortest possible length m over U with value x_{i_1}.

First we introduce some notations:

$$w_1(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \;=\; x_{i_1} x_{i_2} \ldots x_{i_n} \;=\; u_1(x_1, x_2, \ldots, x_n)$$

and

$$w_2(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \;=\; x_1x_2\ldots x_n \;=\; u_2(x_1, x_2, \ldots, x_n) \qquad (2)$$

CASE (i): $i_1 = 1$. Now

$$\begin{aligned}
x_{i_1}x_{i_2}\ldots x_{i_n} &= y_m a_{2m} x_{i_2}\ldots x_{i_n} \quad \text{(from equations (1.2))} \\
&= y_m w_1(a_{2m}, x_{i_2}, \ldots, x_{i_n}) \\
&= y_m w_2(a_{2m}, x_{i_2}, \ldots, x_{i_n}) \quad \text{(since } U \text{ satisfies (1))} \\
&= y_m a_{2m} x_2 \ldots x_n \\
&= x_1 x_2 \ldots x_n \quad \text{(from equations (1.2))}
\end{aligned}$$

as required.

CASE (ii): $i \le i_i < n$. Now, putting $j = i_i$, we have

$$\begin{aligned}
x_{i_1}x_{i_2}\ldots x_{i_n} &= y_m a_{2m} x_{i_2}\ldots x_{i_n} \quad \text{(from equations (1.2))} \\
&= y_m w_1(a_{2m}, x_{i_2}, \ldots, x_{i_n}) \\
&= y_m w_2(a_{2m}, x_{i_2}, \ldots, x_{i_n}) \quad \text{(since } U \text{ satisfies (1))} \\
&= y_m x_1 x_2 \ldots x_{j-1} a_{2m} x_{j+1} \ldots x_n \\
&= y_m x_1 x_2 \ldots x_{j-1} a_{2m-1} t_m z \quad \text{(from equations (1.2)} \\
&\qquad\qquad \text{where } z = x_{j+1}\ldots x_n\text{)} \\
&= y_m x_1 x_2 \ldots x_{j-1} a_{2m-1} b^{(m)}_{j+1} \ldots b^{(m)}_n t'_m z \\
&\qquad \text{(by Result 1.2.8, for some } b^{(m)}_{j+1}, \ldots, b^{(m)}_n \in U \\
&\qquad \text{and } t'_m \in S\backslash U \text{, since } t_m \in S\backslash U\text{)} \\
&= y_m u_2(x_1, x_2, \ldots, x_{j-1},\ a_{2m-1},\ b^{(m)}_{j+1}, \ldots, b^{(m)}_n) t'_m z \\
&= y_m u_1(x_1, x_2, \ldots, x_{j-1},\ a_{2m-1},\ b^{(m)}_{j+1}, \ldots, b^{(m)}_n) t'_m z \qquad (3) \\
&\qquad \text{(since } U \text{ satisfies (1)).}
\end{aligned}$$

Now $u_1(z_1, z_2, \ldots, z_n)$ begins with $z_{i_1} = z_j$, so the product (3) in S contains $y_m a_{2m-1}$ which equals $y_{m-1}a_{2m-2}$ (from equations (1.2)).

Thus the product (3) above equals

$$\begin{aligned}
& y_{m-1}u_1(x_1, x_2, \dots, x_{j-1}, a_{2m-2}, b_{j+1}^{(m)}, \dots, b_n^{(m)})t'_m z \\
= \ & y_{m-1}u_2(x_1, x_2, \dots, x_{j-1}, a_{2m-2}, b_{j+1}^{(m)}, \dots, b_n^{(m)})t'_m z \\
& \qquad\qquad\qquad\qquad \text{(since } U \text{ satisfies (1))} \\
= \ & y_{m-1}x_1x_2 \dots x_{j-1}, a_{2m-2}, b_{j+1}^{(m)}, \dots, b_n^{(m)})t'_m z \\
= \ & y_{m-1}x_1x_2 \dots x_{j-1}a_{2m-2}b_{j+1}^{(m)} \dots b_n^{(m)})t'_m z \quad \text{(since } t_m = b_{j+1}^{(m)} \dots b_n^{(m)}t'_m) \\
= \ & y_{m-1}x_1x_2 \dots x_{j-1}a_{2m-3}t_{m-1}z \quad \text{(from equations (1.2))} \\
& \vdots \\
= \ & y_1x_1x_2 \dots x_{j-1}a_1b_{j+1}^{(1)} \dots t'_1 z \\
& \qquad\qquad \text{(by Result 1.2.8 for some } b_{j+1}^{(1)}, \dots, b_n^{(1)} \in U \\
& \qquad\qquad \text{and } t'_1 S\backslash U, \text{ since } t_1 \in S\backslash U) \\
= \ & y_1u_2(x_1, x_2, \dots, x_{j-1}, a_1, b_{j+1}^{(1)}, \dots, b_n^{(1)})t'_1 z \\
= \ & y_1u_1(x_1, x_2, \dots, x_{j-1}, a_1, b_{j+1}^{(1)}, \dots, b_n^{(1)})t'_1 z \qquad (4) \\
& \qquad\qquad \text{(Since } U \text{ satisfies (1)).}
\end{aligned}$$

Again as before, product (4) in S contains y_1a_1 which equals a_0 (from equations (1.2)). Thus the product (4) above equals

$$\begin{aligned}
& u_1(x_1, x_2, \dots, x_{j-1}, a_0, b_{j+1}^{(1)}, \dots, b_n^{(1)})t'_1 z \\
= \ & u_2(x_1, x_2, \dots, x_{j-1}, a_0, b_{j+1}^{(1)}, \dots, b_n^{(1)})t'_1 z \quad \text{(since } U \text{ satisfies (1))} \\
= \ & x_1x_2 \dots x_{j-1} \ a_0b_{j+1}^{(1)} \dots b_n^{(1)}t' z \\
= \ & x_1x_2 \dots x_{j-1} \ a_0t_1z \quad \text{(since } t_1 = b_{j+1}^{(1)} \dots b_n^{(1)}t'_1) \\
= \ & x_1x_2 \dots x_n, \quad \text{(since } a_0t_1 = x_{i_1} = x_j, \text{ and } z = x_{j+1} \dots x_n)
\end{aligned}$$

which proves the result for $k = 1$ in Case (ii).

CASE (iii): $i_1 = n$. Now

$$x_{i_1}x_{i_2} \dots x_{i_n}$$

$$
\begin{aligned}
&= y_m a_m x_{i_2} \dots x_{i_n} \quad \text{(from equations (1.2))} \\
&= y_m x_1 x_2 \dots x_{n-1} a_{2m} \quad \text{(since } U \text{ satisfies (1))} \\
&= y_m a_m x_1 x_2 \dots x_{n-1} a_{2m-1} t_m \quad \text{(from equations (1.2))} \\
&= y_m a_{2m-1} x_{i_2} \dots x_{i_n} t_m \quad \text{(since } U \text{ satisfies (1))} \\
&= y_{m-1} a_{2m-2} x_{i_2} \dots x_{i_n} t_m \quad \text{(from equations (1.2))} \\
&= y_{m-1} x_1 x_2 \dots x_{n-1} a_{2m-2} t_m \quad \text{(since } U \text{ satisfies (1))} \\
&= y_{m-1} x_1 x_2 \dots x_{n-1} a_{2m-3} t_{m-1} \quad \text{(from equations (1.2))} \\
&\quad \vdots \\
&= y_1 x_1 x_2 \dots x_{n-1} a_1 t_1 \\
&= y_1 a_1 x_{i_2} \dots x_{i_n} t_1 \quad \text{(since } U \text{ satisfies (1))} \\
&= a_0 x_{i_2} \dots x_{i_n} t_1 \quad \text{(from equations (1.2))} \\
&= x_1 x_2 \dots x_{n-1} a_0 t_1 \quad \text{(since } U \text{ satisfies (1))} \\
&= x_1 x_2 \dots x_n \quad \text{(from equations (1.2), since } i_1 = n)
\end{aligned}
$$

as required.

REMARK 1: A proof for Case (iii) could also be obtained from the proof for Case (ii) above by making the following conventions:

(a) the word $x_{j+1} \dots x_n = 1$;

(b) $b_{j+1}^{(k)} = \dots = b_n^{(k)} = 1$ and $t'_k = t_k$ for $k = 1, 2, \dots, m$;

(c) the vector

$$(x_1, x_2, \dots, x_{j-1},\ a_{2k-1}, b_{j+1}^{(k)}, \dots, b_n^{(k)})$$

$$= (x_1, x_2, \dots, x_{n-1},\ a_{2k-1}) \ \text{ for } k = 1, 2, \dots, m$$

(d) the vector

$$(x_1, x_2, \dots, x_{j-1},\ a_{2k-2}, b_{j+1}^{(k)}, \dots, b_n^{(k)});$$

$$= (x_1, x_2, \dots, x_{n-1},\ a_{2k-2}) \ \text{ for } k = 1, 2, \dots, m.$$

So assume now that (1) is true for all $x_{i_1}, x_{i_2}, \ldots, x_{i_{q-1}} \in S$ and all $x_{i_q}, x_{i_{q+1}}, \ldots, x_{i_n} \in U$. We prove from this assumption that (1) is true for all $x_{i_1}, x_{i_2}, \ldots, x_{i_q} \in S$ and for all $x_{i_{q+1}}, x_{i_{q+2}}, \ldots, x_{i_n} \in U$. We need not consider the case where $x_{i_q} \in U$, so we assume that $x_{i_q} \in S \backslash U$. As $x_{i_q} \in S \backslash U$ and $\mathrm{Dom}(U, S) = S$, by Result 1.2.5, we may let (1.2) be a zigzag of shortest possible length m over U with value x_{i_q}.

Put $j = i_q$ and $l = i_{q-1}$.

CASE (i). $l = j - 1$. Now

$$\begin{aligned}
x_{i_1} x_{i_2} \ldots x_{i_n} &= x_{i_1} x_{i_2} \ldots x_{i_{q-1}} (y_m a_{2m}) x_{i_{q+1}} \ldots x_{i_n} \quad \text{(from equations (1.2))} \\
&= x_{i_1} x_{i_2} \ldots (x_{i_{q-1}} y_m) a_{2m} x_{i_{q+1}} \ldots x_{i_n} \\
&= x_1 x_2 \ldots x_{j-2} (x_{i_{q-1}} y_m) a_{2m} x_{j+1} \ldots x_n \quad \text{(by the inductive hypothesis)} \\
&= x_1 x_2 \ldots x_{j-2} x_{i_{q-1}} (y_m a_{2m}) x_{j+1} \ldots x_n \\
&= x_1 x_2 \ldots x_{j-2} x_{j-1} x_j x_{j+1} \ldots x_n \quad \text{(since } x_{i_q} = x_j \text{ and } x_{i_{q-1}} = x_{j-1})
\end{aligned}$$

as required.

CASE (ii). $l < j - 1$ and $j < n$. Now

$$\begin{aligned}
x_{i_1} \ldots x_{i_{q-1}} x_{i_q} \ldots x_{i_n} &= x_{i_1} \ldots x_{i_{q-1}} y_m a_{2m} \ldots x_{i_n} \quad \text{(from equations (1.2))} \\
&= w_1(x_{i_1}, x_{i_2}, \ldots, x_{i_{q-1}}, y_m, a_{2m}, \ldots, x_{i_n}) \\
&= w_2(x_{i_1}, x_{i_2}, \ldots, x_{i_{q-1}}, y_m, a_{2m}, \ldots, x_{i_n}) \\
&\qquad \text{(by the inductive hyposthesis)} \\
&= w_2(x_{i_1}, \ldots, x_{i_{q-1}}, y_m, a_{2m-1} t_m, \ldots, x_{i_n}) \quad \text{(from equations (1.2))} \\
&= x_1, x_2, \ldots, x_{l-1} (x_{i_{q-1}} y_m) x_{l+1}, \ldots, x_{j-1} (a_{2m-1} t_m) z \\
&\qquad \text{(where } z = x_{j+1} \ldots x_n) \\
&= x_1, x_2, \ldots, x_{l-1} (x_{i_{q-1}} y_m) x_{l+1}, \ldots, x_{j-1} a_{2m-1} b_{j+1}^{(m)} \ldots b_n^{(m)} t'_m z \\
&\qquad \text{(by Result 1.2.8 for some } b_{j+1}^{(m)}, \ldots, b_n^{(m)} \in U \\
&\qquad \text{and } t'_m \in S \backslash U, \text{ since } t_m \in S \backslash U \\
&= u_2(x_1, x_2, \ldots, x_{l-1}, x_{i_{q-1}} y_m, x_{l+1}, \ldots, x_{j-1}, a_{2m-1}, b_{j+1}^{(m)}, \ldots, b_n^{(m)}) t'_m z
\end{aligned}$$

$$= u_1(x_1, x_2, \ldots, x_{l-1}, x_{i_{q-1}}y_m, x_{l+1}, \ldots, x_{j-1}, a_{2m-1}, b_{j+1}^{(m)}, \ldots, b_n^{(m)})t'_m z$$

(by the inductive hypothesis) (5)

Since $u_1(z_1, z_2, \ldots, z_n)$ contains as a subword $z_{i_{q-1}}z_{i_q}$, the product (5) in S contains $(x_{i_{q-1}}y_m)a_{2m-1}$ which equals $(x_{i_{q-1}}y_{m-1})a_{2m-2}$ (from equations (1.2)). Thus the product (5) above equals

$$u_1(x_1, \ldots, x_{l-1}, x_{i_{q-1}}y_{m-1}, x_{l+1}, \ldots, x_{j-1}, a_{2m-2}, b_{j+1}^{(m)}, \ldots, b_n^{(m)})t'_m z$$

$$= u_2(x_1, \ldots, x_{l-1}, x_{i_{q-1}}y_{m-1}, x_{l+1}, \ldots, x_{j-1}, a_{2m-2}, b_{j+1}^{(m)}, \ldots, b_n^{(m)})t'_m z$$

(by the inductive hypothesis)

$$= x_1, x_2, \ldots, x_{l-1}, x_{i_{q-1}}y_{m-1}, x_{l+1}, \ldots, x_{j-1}, a_{2m-2}, b_{j+1}^{(m)}, \ldots, b_n^{(m)}t'_m z$$

$$= x_1x_2 \ldots x_{l-1}x_{i_{q-1}}y_{m-1}x_{l+1} \ldots x_{j-1}a_{2m-2}t_m z$$

(since $t_m = b_{j+1}^{(m)} \ldots b_n^{(m)}t'_m$)

$$= x_1x_2 \ldots x_{l-1}x_{i_{q-1}}y_{m-1}x_{l+1} \ldots x_{j-1}a_{2m-3}t_{m-1}z \quad \text{(from equations (1.2))}$$

$$\vdots$$

$$= x_1x_2 \ldots x_{l-1}x_{i_{q-1}}y_1x_{l+1} \ldots x_{j-1}a_1t_1z$$

$$= x_1x_2 \ldots x_{l-1}x_{i_{q-1}}y_1x_{l+1} \ldots x_{j-1}a_1b_{j+1}^{(1)}t'_1z$$

((by Result 1.2.8 for some $b_{j+1}^{(1)}, \ldots, b_n^{(1)} \in U$, $t'_1 \in S\backslash U$, since $t_1 \in S\backslash U$)

$$= u_2(x_1, \ldots, x_{l-1}, x_{i_{q-1}}y_1, x_{l+1}, \ldots, x_{j-1}, a_1, b_{j+1}^{(1)}, \ldots, b_n^{(1)})t'_1z$$

$$= u_1(x_1, \ldots, x_{l-1}, x_{i_{q-1}}y_1, x_{l+1}, \ldots, x_{j-1}, a_1, b_{j+1}^{(m)}, \ldots, b_n^{(1)})t'_1z$$

(by the inductive hypothesis) (6)

Now as $u_1(z_1, z_2, \ldots, z_n)$ contains $z_{i_{q-1}}z_{i_q}$ as a subword, the product (6) in S contains $(x_{i_{q-1}}y_1)a_1$ which equals to $x_{i_{q-1}}a_0$ (from equations (1.2)), Thus the product (6) above equals

$$u_1(x_1, x_2, \ldots, x_{l-1}, x_{i_{q-1}}, x_{l+1}, \ldots, x_{j-1}, a_0, b_{j+1}^{(1)}, \ldots, b_n^{(1)})t_1' z$$

$$= u_2(x_1, x_2, \ldots, x_{l-1}, x_{i_{q-1}} x_{l+1}, \ldots, x_{j-1}, a_0, b_{j+1}^{(1)}, \ldots, b_n^{(1)})t_1' z$$

(by the inductive hypothesis)

$$= x_1 x_2 \ldots x_{l-1} x_{i_{q-1}} x_{l+1} \ldots x_{j-1} a_0 b_{j+1}^{(1)} \ldots b_n^{(1)} t_1' z$$

$$= x_1 x_2 \ldots x_{l-1} x_{i_{q-1}} x_{l+1} \ldots x_{j-1} a_0 t_1 x_{j+1} \ldots x_n$$

(since $z = x_{j+1} \ldots x_n$ and $t_1 = b_{j+1}^{(1)} \ldots b_n^{(1)} t_1'$)

$$= x_1 x_2 \ldots x_n \quad (\text{since } x_{i_{q-1}} = x_l \text{ and } a_0 t_1 = x_{i_q} = x_j)$$

as required.

CASE (iii). $l < j - 1$ and $j = n$. Now

$$x_{i_1} x_{i_2} \ldots x_{i_{q-1}}\, x_{i_q} \ldots x_{i_n}$$

$$= x_{i_1} x_{i_2} \ldots x_{i_{q-1}}\, y_m a_{2m} \ldots x_{i_n} \quad (\text{from equations (1.2)})$$

$$= x_{i_1} x_{i_2} \ldots (x_{i_{q-1}}\, y_m) a_{2m} \ldots x_{i_n}$$

$$= x_1 x_2 \ldots x_{l-1} (x_{i_{q-1}}\, y_m) x_{l+1} \ldots x_{n-1} a_{2m} \quad (\text{by the inductive hypothesis})$$

$$= x_1 x_2 \ldots x_{l-1} (x_{i_{q-1}}\, y_m) x_{l+1} \ldots x_{n-1} a_{2m-1} t_m \quad (\text{from equations (1.2)})$$

$$= x_{i_1} x_{i_2} \ldots (x_{i_{q-1}}\, y_m) a_{2m} \ldots x_{i_n} t_m \quad (\text{by the inductive hypothesis})$$

$$= x_{i_1} x_{i_2} \ldots x_{i_{q-1}} (y_m a_{2m-1}) \ldots x_{i_n} t_m$$

$$= x_{i_1} x_{i_2} \ldots x_{i_{q-1}} (y_{m-1} a_{2m-2}) \ldots x_{i_n} t_m \quad (\text{from equations (1.2)})$$

$$= x_{i_1} x_{i_2} \ldots x_{i_{q-1}} (y_{m-1} a_{2m-2}) \ldots x_{i_n} t_m$$

$$= x_1 x_2 \ldots x_{l-1} (x_{i_{q-1}}\, y_{m-1}) x_{l+1} \ldots x_{n-1} a_{2m-2} t_m \quad (\text{by the inductive hypothesis})$$

$$= x_1 x_2 \ldots x_{l-1} (x_{i_{q-1}}\, y_{m-1}) x_{l+1} \ldots x_{n-1} a_{2m-3} t_{m-1} \quad (\text{from equations (1.2)})$$

$$\vdots$$

$$= x_1 x_2 \ldots x_{l-1} (x_{i_{q-1}}\, y_1) x_{l+1} \ldots x_{n-1} a_1 t_1$$

$$= x_{i_1}x_{i_2}\dots(x_{i_{q-1}}\,y_1)a_1\dots x_{i_n}t_1 \quad \text{(by the inductive hypothesis)}$$

$$= x_{i_1}x_{i_2}\dots x_{i_{q-1}}(y_1a_1)\dots x_{i_n}t_1$$

$$= x_{i_1}x_{i_2}\dots x_{i_{q-1}}a_0\dots x_{i_n}t_1 \quad \text{(from equations (1.2))}$$

$$= x_1x_2\dots x_{l-1}\,x_{i_{q-1}}\,x_{l+1}\dots x_{n-1}a_0t_1 \quad \text{(by the inductive hypothesis)}$$

$$= x_1x_2\dots x_n \quad \text{(from equations (1.2))},$$

since $i_q = n$ and $i_{q-1} = l$ as required.

REMARK 2: A proof for the Case (iii) could also be obtained from the proof for Case (ii) above by making the following conventions:

(a) the word $x_{j+1}\dots x_n = 1$;

(b) $b_{j+1}^{(k)} = \dots = b_n^{(k)} = 1$ and $t'_k = t_k$ for $k = 1, 2, \dots, m$;

(c) the vector

$$(x_1, x_2, \dots, x_{i_{q-1}}\,y_k,\, x_{l+1}, \dots, x_{j-1},\, a_{2k-1}, b_{j+1}^{(k)}, \dots, b_n^{(k)})$$
$$= (x_1, x_2, \dots, x_{i_{q-1}}\,y_k, \dots, x_{n-1}, a_{2k-1}) \quad \text{for } k = 1, 2, \dots, m;$$

and

$$(x_1, x_2, \dots, x_{i_{q-1}}\,y_{k-1},\, x_{l+1}, \dots, x_{j-1},\, a_{2k-2}, b_{j+1}^{(k)}, \dots, b_n^{(k)})$$
$$= (x_1, x_2, \dots, x_{i_{q-1}}\,y_{k-1}, \dots, x_{n-1}, a_{2k-2}) \quad \text{for } k = 1, 2, \dots, m$$

and where $y_0 = 1$.

CASE (iv): $j + 1 < l < n$. We have

$$\begin{aligned}
x_{i_1}x_{i_2}\dots x_{i_n} &= x_{i_1}x_{i_2}\dots x_{i_{q-1}}y_m a_{2m} x_{i_{q+1}}\dots x_{i_n} \quad \text{(from equations (1.2))}\\
&= w_1(x_{i_1}, x_{i_2}, \dots, x_{i_{q-1}}y_m, a_{2m}, x_{i_{q+1}}, \dots, x_{i_n})\\
&= w_2(x_{i_1}, x_{i_2}, \dots, x_{i_{q-1}}y_m, a_{2m}, x_{i_{q+1}}, \dots, x_{i_n})\\
&\qquad \text{(by the inductive hypothesis)}\\
&= x_1x_2\dots x_{j-1}a_{2m}x_{j+1}\dots x_{l-1}x_{i_{q-1}}y_m x_{l+1}\dots x_n\\
&= x_1x_2\dots x_{j-1}a_{2m-1}t_m x_{j+1}\dots x_{l-1}x_{i_{q-1}}y_m x_{l+1}\dots x_n\\
&\qquad \text{(from equations (1.2))}
\end{aligned}$$

$$= \quad x_1x_2\dots x_{j-1}a_{2m-1}b^{(m)}_{j+1}\dots b^{(m)}_{j+(l-j-1)}t'_mx_{j+1}\dots x_{l-1}x_{i_{q-1}}y_mx_{l+1}\dots x_n$$

(by Result 1.2.8 for some $b^{(m)}_{j+1},\dots,b^{(m)}_{j+(l-j+1)}\in U$ and $t'_m\in S\backslash U$, since $t_m\in S\backslash U$)

$$= \quad u_2(x_1,x_2,\dots,x_{j-1},a_{2m-1},b^{(m)}_{j+1},\dots,b^{(m)}_{j+(l-j-1)},t'_mx_{j+1}\dots x_{i_{q-1}}y_m,x_{l+1},\dots,x_n)$$

$$= \quad u_1(x_1,x_2,\dots,x_{j-1},a_{2m-1},b^{(m)}_{j+1},\dots,b^{(m)}_{j+(l-j-1)},t'_mx_{j+1}\dots x_{i_{q-1}}y_m,x_{l+1},\dots,x_n)$$

(by the inductive hypothesis) (7)

Now since the word $u_1(z_1,z_2,\dots,z_n)$ contains $z_{i_{q-1}}z_{i_q}$ as a subword, the product (7) in S contains $(x_{i_{q-1}}y_m)a_{2m-1}$ which equals $(x_{i_{q-1}}y_{m-1})a_{2m-2}$ (from equations (1.2)). Thus the product (7) above equals

$$u_1(x_1,x_2,\dots,x_{j-1},a_{2m-2},b^{(m)}_{j+1},\dots,b^{(m)}_{j+(l-j-1)},t'_mx_{j+1}\dots x_{l-1}x_{i_{q-1}}y_{m-1},x_{l+1},\dots,x_n)$$

$$= \quad u_2(x_1,x_2,\dots,x_{j-1},a_{2m-2},b^{(m)}_{j+1},\dots,b^{(m)}_{j+(l-j-1)},t'_mx_{j+1}\dots x_{l-1}x_{i_{q-1}}y_{m-1},x_{l+1},\dots,x_n)$$

(by the inductive hypothesis)

$$= \quad x_1x_2\dots x_{j-1}a_{2m-2}b^{(m)}_{j+1}\dots b^{(m)}_{j+(l-j-1)}t'_mx_{j+1}\dots x_{l-1}x_{i_{q-1}}y_{m-1}x_{l+1}\dots x_n)$$

$$= \quad x_1x_2\dots x_{j-1}a_{2m-2}t_mx_{j+1}\dots x_{l-1}x_{i_{q-1}}y_{m-1}x_{l+1}\dots x_n$$

(since $t_m=b^{(m)}_{j+1}\dots b^{(m)}_{j+(l-j-1)}t'_m$)

$$= \quad x_1x_2\dots x_{j-1}a_{2m-3}t_{m-1}x_{j+1}\dots x_{l-1}x_{i_{q-1}}y_{m-1}\;x_{l+1}\dots x_n \text{ (from equations (1.2))}$$

$$\vdots$$

$$= \quad x_1x_2\dots x_{j-1}a_1t_1x_{j+1}\dots x_{l-1}x_{i_{q-1}}y_1\;x_{l+1}\dots x_n$$

$$= \quad x_1x_2\dots x_{j-1}a_1b^{(1)}_{j+1}\dots b^{(1)}_{j+(l-j-1)}t'_1x_{j+1}\dots x_{l-1}x_{i_{q-1}}y_1\;x_{l+1}\dots x_n$$

(by Result 1.2.8 for some $b^{(1)}_{j+1},\dots,b^{(1)}_{j+(l-j-1)}\in U$ and $t'_1\in S\backslash U$, since $t_1\in S\backslash U$)

$$= \quad u_2(x_1,x_2,\dots,x_{j-1},a_1,b^{(1)}_{j+1},\dots,b^{(1)}_{j+(l-j-1)},t'_1x_{j+1}\dots x_{i_{q-1}}y_1,x_{l+1},\dots,x_n)$$

$$= \quad u_1(x_1,x_2,\dots,x_{j-1},a_1,b^{(1)}_{j+1},\dots,b^{(1)}_{j+(l-j-1)},t'_1x_{j+1}\dots x_{i_{q-1}}y_1,x_{l+1},\dots,x_n)$$

(by the inductive hypothesis) (8)

As $u_1(z_1, z_2, \ldots, z_n)$ contains $z_{i_{q-1}} z_{i_q}$ as a subword, the product (8) in S contains $(x_{i_{q-1}} y_1) a_1$ which equals $x_{i_{q-1}} a_0$ (from equations (1.2)). Thus the product (8) above equals

$$
\begin{aligned}
& u_1(x_1, x_2, \ldots, x_{j-1}, a_0, b_{j+1}^{(1)}, \ldots, b_{j+(l-j-1)}^{(1)}, t_1' x_{j+1} \ldots x_{i_{q-1}}, x_{l+1}, \ldots, x_n) \\
= {} & u_2(x_1, x_2, \ldots, x_{j-1}, a_0, b_{j+1}^{(1)}, \ldots, b_{j+(l-j-1)}^{(1)}, t_1' x_{j+1} \ldots x_{i_{q-1}}, x_{l+1}, \ldots, x_n) \\
& \qquad \text{(by the inductive hypothesis)} \\
= {} & x_1 x_2 \ldots x_{j-1} a_0 b_{j+1}^{(1)} \ldots b_{j+(l-j-1)}^{(1)} t_1' x_{j+1} \ldots x_{l-1} x_{i_{q-1}} x_{l+1} \ldots x_n \\
= {} & x_1 x_2 \ldots x_{j-1} a_0 t_1 x_{j+1} \ldots x_{l-1} x_{i_{q-1}} x_{l+1} \ldots x_n \\
= {} & x_1 x_2 \ldots x_n \qquad \text{(since } x_{i_{q-1}} = x_l \text{ and } x_{i_q} = a_0 t_1 = x_j\text{)}
\end{aligned}
$$

as required.

CASE (v): $j + 1 = l$. Now

$$
\begin{aligned}
x_{i_1} x_{i_2} \ldots x_{i_n} &= x_{i_1} x_{i_2} \ldots x_{i_{q-1}} a_{2m} x_{i_n} \quad \text{(from equations (1.2))} \\
&= x_{i_1} x_{i_2} \ldots (x_{i_{q-1}} y_m) a_{2m} \ldots x_{i_n}) \\
&= x_1, x_2 \ldots x_{j-1} a_{2m} (x_{i_{q-1}} y_m) x_{l+1} \ldots x_n) \\
&\quad \text{(by the inductive hypothesis; if } l = n \text{, the product } x_{l+1} \ldots x_n = 1\text{)} \\
&= x_1 x_2 \ldots x_{j-1} a_{2m-1} t_m (x_{i_{q-1}} y_m) x_{l+1} \ldots x_n \\
&= x_1 x_2 \ldots x_{j-1} a_{2m-1} (t_m x_{i_{q-1}} y_m) x_{l+1} \ldots x_n \\
&= x_{i_1} x_{i_2} \ldots (x_{i_{q-2}} (t_m x_{i_{q-1}} y_m) a_{2m-1} x_{i_{q+1}} \ldots x_{i_n}) \\
&\quad \text{(by the inductive hypothesis)} \\
&= x_{i_1} x_{i_2} \ldots (x_{i_{q-2}} (t_m x_{i_{q-1}} y_{m-1}) a_{2m-2} x_{i_{q+1}} \ldots x_{i_n}) \quad \text{(from equations (1.2))} \\
&= x_1 x_2 \ldots x_{j-1} a_{2m-2} (t_m x_{i_{q-1}} y_{m-1}) x_{l+1} \ldots x_n \\
&\quad \text{(by the inductive hypothesis)} \\
&= x_1 x_2 \ldots x_{j-1} a_{2m-3} (t_{m-1} x_{i_{q-1}} y_{m-1}) x_{l+1} \ldots x_n \quad \text{(from equations (1.2))} \\
&\ \ \vdots \\
&= x_1 x_2 \ldots x_{j-1} a_1 (t_1 x_{i_{q-1}} y_1) x_{l+1} \ldots x_n \\
&= x_{i_1} x_{i_2} \ldots (x_{i_{q-2}} (t_1 x_{i_{q-1}} y_1) a_1 x_{i_{q+1}} \ldots x_{i_n}) \text{(by the inductive hyposthesis)}
\end{aligned}
$$

$$= x_{i_1}x_{i_2}\dots(x_{i_{q-2}}(t_1x_{i_{q-1}})a_0x_{i_{q+1}}\dots x_{i_n}) \text{ (from equations (1.2))}$$

$$= x_1x_2\dots x_{j-1}a_0(t_1x_{i_{q-1}})x_{l+1}\dots x_n \text{ (by the inductive hyposthesis)}$$

$$= x_1x_2\dots x_n \text{ (from equations (1.2) and since } i_{q-1}=l=j+1)$$

as required.

Finally, a proof in the remaining Case (vi), namely when $j+1<l$ and $l=n$ can be obtained from the proof for Case (iv) above by making the following conventions:

(a) the word $x_{l+1}\dots x_n = 1$;

(b) the vector

$$(x_1, x_2, \dots, x_{j-1}, a_{2k-1}, b^{(k)}_{j+1}, \dots, b^{(k)}_{j+(l-j-1)}, t'_kx_{j+1}\dots x_{l-1}x_{i_{q-1}}y_k, x_{l+1}, \dots, x_n)$$
$$= (x_1, x_2, \dots, x_{j-1}, a_{2k-1}, b^{(k)}_{j+1}, \dots, b^{(k)}_{n-1}, t'_kx_{j+1}\dots x_{n-1}x_{i_{q-1}}y_k)$$

for $k=1,2,\dots,m$

(b) the vector

$$(x_1, x_2, \dots, x_{j-1}, a_{2k-2}, b^{(k)}_{j+1}, \dots, b^{(k)}_{j+(l-j-1)}, t'_kx_{j+1}\dots x_{l-1}x_{i_{q-1}}y_{k-1}, x_{l+1}, \dots, x_n)$$
$$= (x_1, x_2, \dots, x_{j-1}, a_{2k-2}, b^{(k)}_{j+1}, \dots, b^{(k)}_{n-1}, t'_kx_{j+1}\dots x_{n-1}x_{i_{q-1}}y_{k-1})$$

for $k=1,2,\dots,m$ and where $y_0=1$. This completes the proof of the Theorem 3.1.

The following corollary gives a sufficient condition for $\mathrm{Dom}(U,S)$ to satisfy any permutation identity that U satisfies and, thus generalizes Result 1.2.10 from commutativity to any permutation identity.

COROLLARY 3.2 (to the proof of Theorem 3.1): Let U and S be any semigroups with U a subsemigroup of S. Let U satisfies a permutation identity (1). If for all $s \in S\backslash U$, $s = as'$ for some $a \in U$ and $s' \in S$, then $\mathrm{Dom}(U,S)$ also satisfies the permutation identity (1) satisfied by U.

REMARK 3: Theorem 3.1 generalizes Result 1.2.10, which implied that commutativity is preserved under epis of semigroups.

EXAMPLE 3.3[30, Example 8.5]: This shows that the nontrivial permutation identities other than commutativity are not carried over to dominions.

Let F_X be the free semigroup on a countable infinite set $X = \{x_1, x_2, \ldots\}$. Let $U = \langle Y \rangle$, the subsemigroup of F_X generated by the set Y, where $Y = \bigcup_{n=0}^{\infty} \{x_{3n+1}x_{3n+2}, x_{3n+2}, x_{3n+2}x_{3n+3}\}$. Put $S = F_X/\rho$ and $\bar{U} = U\rho$, where ρ is the congruence generated by the relation ρ_0 which consists of the pairs $(u_1u_2\ldots u_n, u_{i_1}u_{i_2}\ldots u_{i_n})$ with $u_j \in U$ for $j = 1, 2, \ldots, n$, and where i is a fixed nontrivial permutation of the set $\{1, 2, \ldots, n\}$ with $n \geq 3$. It is easy to see that for each $n = 0, 1, 2, \ldots, (x_{3n+1}x_{3n+2}x_{3n+3})\rho \in \mathrm{Dom}(\bar{U}, S)$. Now we show that $\mathrm{Dom}(\bar{U}, S)$ does not satisfy the permutation identity corresponding to the permutation i.

To see this consider the product $(x_1x_2x_3)(x_4x_5x_6)\ldots(x_{3n+1}x_{3n+2}x_{3n+3})$ in F_X. Since no n members of U occur consecutively in this word no elementary ρ_0 transition is possible from this base and hence $\mathrm{Dom}(\bar{U}, S)$ does not satisfy the permutation identity corresponding to the permutation i.

CHAPTER 4

CONSEQUENCES OF PERMUTATION IDENTITIES AND SATURATED PERMUTATIVE VARIETIES OF SEMIGROUPS

4.1. INTRODUCTION

In this chapter, after proving some consequences of permutation identities about semigroups, we completely determine saturated permutative varieties of semigroups. We then find sufficient conditions on a homotypical identity to ensure that any variety satisfying it is saturated. The contents of this chapter are from [46, 47, 48].

As before, a semigroup S satisfying a nontrivial permutation identity

$$x_1x_2\ldots x_n \;=\; x_{i_1}x_{i_2}\ldots x_{i_n} \tag{1}$$

will be called a permutative semigroup.

First we prove some useful facts about permutative semigroups.

PROPOSITION 4.1.1[46, Proposition 4.1]: Let U be a permutative semigroup and S any semigroup containing U properly such $\mathrm{Dom}(U,S) = S$. Then for any $x,y \in S$ and $s,t \in S\backslash U$,

$$sxyt \;=\; syxt.$$

PROOF: Since U is permutative, by Theorem 3.1, S is also permutative. Therefore, by Theorem 2.2.5, there exists n and $j \in \{1,2,\ldots,n\}$ such that S also satisfies the following identity

$$x_1x_2\ldots x_{j-1}xyx_j\ldots x_n \;=\; x_1x_2\ldots x_{j-1}yxx_j\ldots x_n$$

(when $j=1$, we assume that the word $x_1x_2\ldots x_{j-1}$ is the empty word).

Since, by Result 1.2.8 for all $s,t \in S\backslash U$ we have $s,t \in S^k$ for all positive integers k, the result now follows.

REMARK 1: Proposition 4.1.1 can also be proved by appealing to ([65, Theorem 1]) and Theorem 3.1.

The proof of Proposition 4.1.1 could be easily modified to give the following corollaries. In Corollaries 4.1.3 and 4.1.4, bracketed statements are dual to the other statements.

In Results 4.1.2 to 4.1.7, U is any semigroup satisfying (1), and S is any semigroup containing U properly and such that $\mathrm{Dom}(U, S) = S$.

COROLLARY 4.1.2: If (1) is nontrivial, then

$$sx_1x_2\dots x_kt \;=\; sx_{j_1}, x_{j_2}\dots x_{j_k}t$$

for all $s, t \in S\backslash U$, $x_1, x_2, \dots, x_k \in S$ and for any permutation j of the set $\{1, 2, \dots, k\}$.

COROLLARY 4.1.3: If $i_1 \neq 1$ $[i_n \neq n]$, then

$$xyt \;=\; yxt[sxy \;=\; syx]$$

for all $x, y \in S$ and $t \in S\backslash U$ $[s \in S\backslash U]$.

COROLLARY 4.1.4: If $i_1 \neq 1$ $[i_n \neq n]$, then

$$x_1x_2\dots x_kt \;=\; x_{j_1}x_{j_2}\dots x_{j_k}t \;\; [sx_1x_2\dots x_k \;=\; sx_{j_1}x_{j_2}\dots x_{j_k}],$$

for all $x_1, x_2, \dots, x_k \in S$ and $t \in S\backslash U$ $[s \in S\backslash U]$, and for any permutation j of the set $\{1, 2, \dots, k\}$.

PROPOSITION 4.1.5: If $i_1 \neq 1$ and either $i_{n-1} \neq n-1$ or $i_n \neq n$, then

$$xyz \;=\; yxz$$

for all $x, z \in S$ and $y \in S\backslash U$.

PROOF: Since $y \in S\backslash U$, we may let (1.2) be a zigzag of shortest possible length m over U with value y. Then

$$\begin{aligned} xyz &= xa_0t_1z \text{ (from equations (1.2))} \\ &= a_0xt_1z \text{ (by Corollary 4.1.3, since } i_1 \neq 1 \text{ and } t_1 \in S\backslash U) \\ &= y_1a_1xt_1z \text{ (from equations (1.2))} \\ &= y_1xa_1xt_1z \text{ (by Corollary 4.1.4, since } i_1 \neq 1 \text{ and } t_1 \in S\backslash U) \\ &= y_1xa_2t_2z \text{ (from equations (1.2))} \end{aligned}$$

$= \quad y_1a_2xt_2z$ (by Corollary 4.1.4, since $i_1 \neq 1$ and $t_2 \in S\backslash U$)

$= \quad y_2a_3xt_2z = \ldots = y_ma_{2m-1}xt_mz$

$= \quad y_mxa_{2m-1}xt_mz$ (by Corollary 4.1.4, since $i_1 \neq 1$ and $t_m \in S\backslash U$)
$= \quad y_mxa_{2m}z$ (from equations (1.2))

$= \quad y_ma_{2m}xz$ (by Theorem 2.2.5 if $i_{n-1} \neq n-1$ and $i_n = n$ since $y_m \in S^{n-1}$, otherwise by Corollary 4.1.4 since $y_m \in S\backslash U$)

$= \quad yxz$(from equations (1.2)) as required.

We give a corollary to Proposition 4.1.5 and its dual.

COROLLARY 4.1.6: If $i_1 \neq 1$ and $i_n \neq n$, then

$$s_1s_2s_3 \quad = \quad s_{j_1}s_{j_2}s_{j_3}$$

for any $s_1, s_2, s_3 \in S$ with one or more being in $S\backslash U$, and for any permutation j of the set $\{1, 2, 3\}$.

COROLLARY 4.1.7: If $i_1 \neq 1$ and either $i_{n-1} \neq n-1$ or $i_n \neq n$, then

$$s_1s_2\ldots s_k \quad = \quad s_{j_1}s_{j_2}\ldots s_{j_{k-1}}s_k$$

for any $s_1, s_2, \ldots, s_k \in S$ such that $s_q \in S\backslash U$ for some $q \in \{1, 2, \ldots, k-1\}$ and for any permutation j of the set $\{1, 2, , \ldots, k-1\}$.

PROOF: We have

$$\begin{aligned} s_1s_2\ldots s_q\ldots s_k &= s_1s_2\ldots s_{q-1}s_{q+1}\ldots s_{k-1}s_qs_k \quad \text{(by Proposition 4.1.5)} \\ &= s_{j_1}s_{j_2}\ldots s_{j_{l-1}}s_{j_{l+1}}\ldots s_{j_{k-1}}s_qs_k \quad \text{(where } s_q = b_{j_l} \text{ by Corollary 4.1.4)} \\ &= s_{j_1}s_{j_2}\ldots s_{j_{l-1}}s_qs_{j_{l+1}}\ldots s_{j_{k-1}}s_k \quad \text{(by Proposition 4.1.5)} \\ &= s_{j_1}s_{j_2}\ldots s_{j_{k-1}}s_k \end{aligned}$$

as required.

PROPOSITION 4.1.8: Let U and S be any semigroups with U a subsemigroup of S and such that $\mathrm{Dom}(U, S) = S$. Take any $d \in S\backslash U$. Let (1.2) be a zigzag of length

m over U with value d with $y_1 \in S \backslash U$ (for example if the zigzag is of shortest possible length). If U satisfies any nontrivial permutation identity, then $d^k = a_0^k t_1^k$ for any positive integer k.

PROOF: We have

$$\begin{aligned} d^k &= (a_0 t_1)^k \\ &= a_0 t_1 (a_0 t_1)^{k-2} a_0 t_1 \quad (\text{if } k-2=0,\ (a_0t_1)^{k-2}=1) \\ &= y_1 a_1 t_1 (a_0 t_1)^{k-2} a_0 t_1 \\ &= y_1 a_1 a_0^{k-1} t_1^k \quad (\text{by Corollary 4.1.2, since } y_1, t_1 \in S \backslash U) \\ &= a_0^k t_1^k, \end{aligned}$$

as required.

In the following theorem we get a sufficient condition for any permutative variety to be saturated.

THEOREM 4.1.9[46, Theorem 5.1]: A permutative variety V is saturated if it admits an identity I such that

(i) I is not a permutation identity, and

(ii) at least one side of I has no repeated variable.

PROOF: To prove the theorem, we can assume without loss of generality, which we prove below, that I has the form

$$x_1 x_2 \dots x_m = w(x_1, x_2, \dots, x_m) \tag{2}$$

where $|x_i|_w \geq 1$ for $i = 1, 2, \dots, m$, and $|x_j|_{w'} \geq 2$ for some $j \in \{1, 2, \dots, m\}$ (recall that $|x|_{w'}$ for any variable x, is the number of occurences of the variable x in the word w).

For if I is homotypical, as I is nonpermutative, I has to be of the form (2). So let us assume next that I is not homotypical. Then I has one of the following three forms:

(i) $x_1 x_2 \dots x_m = f(x_1, x_2, \dots, x_m, x_{m+1}, \dots, x_{m+k})$ where f is some word in the variables $x_1, \dots, x_{m+k}$ such that $k > 0$, and $|x_i|_f \geq 1$, $i = 1, 2, \dots, m+k$;

(ii) $x_1x_2\dots x_m = f(x_1, x_2, \dots, x_m)$, where f is some word in the variables $x_1, \dots, x_m$ such that for some $i \in \{1, 2, \dots, m\}$, $|x_j|_f \geq 1$;

(iii) $x_1x_2\dots x_m = f(x_1, x_2, \dots, x_m, , x_{m+1}, \dots, x_{m+k})$, $k > 0$, where f is some word in $x_1, \dots, x_{m+k}$ such that for some $i \in \{1, 2, \dots, m\}$, $|x_i|_f = 0$ and $|x_{m+j}| \geq 1$, for $j = 1, 2, \dots, k$.

Now in Case (i), by replacing the variables $x_{m+1}, \dots, x_{m+k}$ if necessary, by x_j for some $j \in \{1, 2, \dots, m\}$, we immediately get an identity of the form (2).

In Case (ii), let $j \in \{1, 2, \dots, m\}$ be such that $x_j \notin C(f)$ (the content of f). Now by replacing the variable x_j by x_j^2 we get that S satisfies the identity $x_1x_2\dots x_m = x_1\dots x_{j-1}x_j^2x_{j+1}\dots x_m$ (since the R.H.S. of I is independent of the choice of the variable x_j) which is of the form (2) above.
Finally in the last case we can get an identity of the form (2) simply by applying the techniques of the above two cases.

To prove the theorem, let us now assume to the contrary that $\exists U \in V$ such that U is not saturated. Therefore there exists a semigroup S containing U properly such that $\text{Dom}(U, S) = S$.

LEMMA 4.1.10: For all $a \in U$, $x, y \in S\backslash U$, and for some $q \geq 2$

$$xay = xa^qy.$$

PROOF: Since $y \in S\backslash U$, from Result 1.2.8, we have

$$y = a_1a_2\dots a_my_m$$

for some $y_m \in S\backslash U$ and $a_1, a_2, \dots, a_m \in U$. Now

$$\begin{aligned} xay &= xaa_1a_2\dots a_my_m \\ &= xa_1a_2\dots(aa_j)\dots(a_my_m) \ \text{(by Corollary 4.1.2)} \\ &= xw(a_1, a_2, \dots, aa_j, \dots, a_m)y_m \ \text{(since } U \text{ satisfies (2))} \\ &= xa^qw(a_1, a_2, \dots, a_j, \dots, a_m)y_m \ \text{(by Corollary 4.1.2, where } q = |x_j|_w \geq 2\text{)} \\ &= xa^qa_1a_2\dots a_my_m \\ &= xa^qy \end{aligned}$$

as required. We give a corollary to the proof of Lemma 4.1.10

COROLLARY 4.1.11: For all $a \in U$, $s, t \in S^1$ and $x, y \in S \backslash U$,

$$xsaty = xsa^q ty \quad \text{for some } q \geq 2.$$

Now to complete the proof of Theorem 4.1.9, we take any $d \in S \backslash U$, and let (1.2) be a zigzag for d of shortest possible length m over U. Then

$$\begin{aligned}
d &= y_1 a_1 t_1 \\
&= y_1 a_1^q t_1 \quad \text{(by Result 1.2.5 and Lemma 4.1.10)} \\
&= y_1 a_1^{q-1} a_2 t_2 \quad \text{(from equation (1.2))} \\
&= y_1 a_2 a_1^{q-1} t_2 \quad \text{(by Corollary 4.1.2)} \\
&= y_2 a_3 a_1^{q-1} t_2 \quad \text{(from equation (1.2))} \\
&= y_2 a_1^{q-1} a_3 t_2 \quad \text{(by Corollary 4.1.2)} \\
&= y_2 a_1^{q-1} a_3^q t_2 \quad \text{(by Corollary 4.1.2)} \\
&\;\;\vdots \\
&= y_m a_1^{q-1} a_3^{q-1} \ldots a_{2m-3}^{q-1} a_{2m-1}^q t_m \\
&= y_m a_1^{q-1} a_3^{q-1} \ldots a_{2m-1}^{q-1} a_{2m-1} t_m \\
&= y_m a_{2m-1} a_1^{q-1} a_3^{q-1} \ldots a_{2m-3}^{q-1} a_{2m-1}^{q-2} a_{2m-1} t_m \quad \text{(by Corollary 4.1.2)}
\end{aligned}$$

$$\begin{aligned}
&= y_{m-1} a_{2m-2} a_1 a_3 \ldots a_{2m-3} w' a_{2m-1} t_m \quad \text{(from equations (1.2) and by Corollary 4.1.2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{where } w' = a_1^{q-2} a_3^{q-2} \ldots a_{2m-1}^{q-2}) \\
&= y_1 a_2 a_4 \ldots a_{2m-2} a_1 w' a_{2m-1} t_m \\
&= y_1 a_2 a_4 \ldots a_{2m-2} w' a_{2m-1} t_m \quad \text{(by Corollary 4.1.2)} \\
&= a_0 a_2 a_4 \ldots a_{2m-2} w' a_{2m} \in U,
\end{aligned}$$

a contradiction. This completes the proof of the theorem.

The next theorem characterizes all permutative varieties which are saturated, and thus provides a generalization of [?, Theorem 4.5] from commutative varieties to permutative varieties.

THEOREM 4.1.12[46, Theorem 5.4]: A permutative variety V is saturated if and only if it admits an identity I such that

(i) I is not a permutation identity, and

(ii) at least one side of I has no repeated variable.

PROOF: The 'if' statement of the theorem follows from Theorem 4.1.9. So it remains only to show that any saturated permutative variety V has to admit an identity I of the above form. Now take any saturated permutative variety V, and suppose that V does not admit any identity I of the above form. Therefore all the identities of V are either permutation identities or of the type whose both sides have repeated variables. Since the semigroup U of [28] is commutative, it satisfies all identities of V by ([28], Lemma 3), whence $U \in V$. As U is not saturated, V can't be saturated, a contradiction. This completes the proof of the theorem.

REMARK 2: P.M. Higgins has shown with a different technique ([33], Corollary to Theorem 19) that if a variety V admits an identity of the form given in the statement of Theorem 4.1.12 and a permutation identity (1) such that $i_1 \neq 1$ and $i_n \neq n$, then V is saturated.

In the following theorems, we find sufficient conditions on a homotypical identity to ensure that any semigroup satisfying it is saturated. We prove that a semigroup variety admitting an identity either of the following forms

(I) $x_1x_2\ldots x_n = y(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)x_j$ with $j \neq n$ and which is not a permutation identity; or

(II) $x_1x_2\ldots x_n = f(x_1, x_2, \ldots, x_n)$ for some word f in the variables $x_1, x_2, \ldots, x_n$ which is not a permutation identity, such that there are variables $x_i \neq x_j$ with $|x_i|_f = |x_j|_f = 1$ such that x_ix_j is a subword of f and $j \neq i + 1$, is saturated.

DEFINITION 4.1.13: A variety V of semigroups is called homotypical variety of semigroups if it does not admit any heterotypical identity.

THEOREM 4.1.14[45, Theorem 3.4]: Let V be any homotypical variety admitting an identity of the form

$$x_1x_2\dots x_n = g(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n)x_j \tag{3}$$

with $j \neq n$ (or dually, of the form $x_1x_2\dots x_n = x_jg(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ with $j \neq 1$.

Then V admits the permutation identity

$$x_1x_2\dots x_jxyx_{j+1}\dots x_n = x_1x_2\dots x_jyxx_{j+1}\dots x_n. \tag{4}$$

PROOF: To show that V admits (4), we need to show that every $U \in V$ satisfies (4). For this take any $U \in V$ and any

$$x, y, x_1, x_2, \dots, x_n \in U.$$

Now

$$\begin{aligned}
x_1x_2\dots x_jxyx_{j+1}\dots x_n &= x_1x_2\dots(x_jx)(yx_{j+1})\dots x_n \\
&= g(x_1, x_2, \dots, x_{j-1}, yx_{j+1}, \dots, x_n)(x_jx) \quad \text{(since } U \text{ satisfies (3))} \\
&= x_1x_2\dots x_j(yx_{j+1})\dots x_nx \quad \text{(since } U \text{ satisfies (3))} \\
&= x_1x_2\dots(x_jy)x_{j+1}, \dots x_nx \\
&= g(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots x_n)(x_jy)x \quad \text{(since } U \text{ satisfies (3))} \\
&= g(x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots x_n)(x_jyx) \\
&= (x_1x_2\dots x_jyxx_{j+1})\dots x_n \quad \text{(since } U \text{ satisfies (3))}
\end{aligned}$$

as required.

The following corollary gives a sufficient condition for a homotypical variety to be saturated and follows directly from Theorem 4.1.9 and Theorem 4.1.14.

COROLLARY 4.1.15: Let V be any homotypical variety admitting an identity of the form

$$x_1x_2\ldots x_n = g(x_1, x_2, \ldots x_{j-1}, x_{j+1}, \ldots, x_n)x_j$$

with $j \neq n$, and which is not a permutation identity. Then V is saturated.

REMARK 3: If $j = n$, then the variety V in the Corollary 4.1.15 is not necessarily saturated. For example, not all bands are saturated ([31], Corollary 4). P.M. Higgins ([33], Theorem 16) has shown a related result, namely that if a variety V admits a homotypical identity of the form

$$x_1x_2\ldots x_n = f(x_1, x_2, \ldots, x_n)$$

which is not a permutation identity and is such that f neither begins with x_1 nor ends with x_n, then V is saturated.

THEOREM 4.1.16[48, Theorem 3.1]: Let V be any semigroup variety V admitting a homotypical identity

$$I:\ x_1x_2\ldots x_n = f(x_1, x_2, \ldots, x_n) \tag{5}$$

for some word f in the variables $x_1, x_2, \ldots, x_n$. If I is such that

(i) I is not a permutation identity, and

(ii) there are variables $x_i \neq x_j$ with $|x_i|_f = |x_j|_f = 1$ such that x_ix_j is a subword f and $j \neq i+1$,

then V is saturated.

PROOF: By Theorem 4.1.9, it is sufficient to show that V is permutative. This we do by showing that every member of V satisfies some fixed permutation identity. For this take any $U \in V$.

CASE (i). $i < j$. By condition (ii) of the hypothesis, let

$$f(x_1, x_2, \ldots, x_n) = g_1(\hat{x})x_ix_jg_2(\hat{x}) \tag{6}$$

where $\hat{x} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ and g_1, g_2 are some words.

Take any $x, y, x_1, x_2, \ldots, x_n \in U$. Then

$$\begin{aligned}
x_1x_2\ldots x_ixyx_{i+1}x_n &= f(x_1, x_2, \ldots, x_i, xyx_{i+1}, \ldots, x_n) \text{ (since } U \text{ satisfies equation (5))} \\
&= g_1(\hat{x}(y))(x_ix)x_jg_2(\hat{x}(y)) \text{ (from equation (6), where} \\
\hat{x}(b) &= (x_1, x_2, \ldots, x_{i-1}, bx_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \text{ for any } b \in U) \\
&= g_1(\hat{x}(y))x_i(xx_j)g_2(\hat{x}(y)) \\
&= f(x_1, \ldots, x_i, yx_{i+1}, \ldots, xx_j, \ldots, x_n) \text{ (from eq. (5) since } j \neq i+1 \\
&= x_1x_2\ldots x_iyx_{i+1}\ldots xx_j\ldots x_n \text{ (since } U \text{ satisfies equation (5))} \\
&= f(x_1, x_2, \ldots, x_iy, x_{i+1}, \ldots, xx_j, \ldots, x_n) \text{ (since } U \text{ satisfies eq. (5)} \\
&= g_1(\hat{x})(x_iy)(xx_j)g_2(\hat{x}) \text{(from equation (6))} \\
&= g_1(\hat{x})(x_iyx)x_jg_2(\hat{x}) \\
&= f(x_1, x_2, \ldots, x_i, yxx_{i+1}, \ldots, x_n) \text{ (from equation (6))} \\
&= x_1x_2\ldots x_ixyx_{i+1}\ldots x_n \text{ (since } U \text{ satisfies equation (5))}
\end{aligned}$$

whence U satisfies the permutation identity

$$x_1x_2\ldots x_{n+2} = x_1x_2\ldots x_ix_{i+2}x_{i+1}x_{i+3}\ldots x_{n+2}.$$

Hence U is permutative.

CASE (ii). $j < i$. By the condition (ii) of the hypothesis, let

$$f(x_1, x_2, \ldots, x_n) = h_1(\bar{x})x_ix_jh_2(\bar{x}) \tag{7}$$

where $\bar{x} = (x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and h_1, h_2 are some words.

Take any $x, y, x_1, x_2, \ldots, x_n \in U$.

CASE (ii)(a). $j = 1$. Now

$$\begin{aligned}
xyx_1x_2\ldots x_n &= xf(yx_1, x_2, \ldots, x_n) \text{ (since } U \text{ satisfies equation (5))} \\
&= xh_1(\bar{x})x_i(yx_1)h_2(\bar{x}) \text{ (from equation (7)} \\
&= xh_1(\bar{x})(x_iy)x_1h_2(\bar{x})
\end{aligned}$$

$$
\begin{aligned}
&= xf(x_1, x_2, \ldots, x_iy, \ldots, x_n) \quad \text{(from equation (7)} \\
&= xx_1x_2, \ldots, x_iy \ldots x_n \quad \text{(since } U \text{ satisfies equation (5))} \\
&= f(xx_1, x_2, \ldots, x_iy, \ldots, x_n) \quad \text{(since } U \text{ satisfies equation (5))} \\
&= h_1(\bar{x})(x_iy)(xx_1)h_2(\bar{x}) \quad \text{(from equation (7)} \\
&= h_1(\bar{x})x_i(yxx_1)h_2(\bar{x}) \\
&= f(yxx_1, x_2, \ldots, x_n) \quad \text{(from equation (7))} \\
&= yxx_1x_2 \ldots x_n \quad \text{(since } U \text{ satisfies equation (5))}.
\end{aligned}
$$

Thus U satisfies the permutation identity $x_1x_2 \ldots x_{n+2} = x_2x_1x_3 \ldots x_{n+2}$ and so U is permutative.

CASE (ii)(b). $1 < j < n$. Now

$$
\begin{aligned}
x_1x_2 \ldots x_{j-1}xyx_j \ldots x_n &= f(x_1, x_2, \ldots, x_{j-1}x, yx_j, \ldots, x_n) \\
&\qquad\qquad \text{(since } U \text{ satisfies equation (5))} \\
&= h_1(\bar{x}(x))x_i(yx_j)h_2(\bar{x}(x)) \quad \text{(from equation (7) where} \\
\bar{x}(b) &= (x_1, \ldots, x_{j-1}b, x_{j+1}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \quad \text{(for any } b \in U\text{)} \\
&= h_1(\bar{x}(x))(x_iy)x_jh_2(\bar{x}(x)) \\
&= f(x_1, \ldots, x_{j-1}x, x_j, \ldots, x_iy, \ldots, x_n) \quad \text{(from equation (7)} \\
&= x_1x_2 \ldots x_{j-1}xx_j \ldots x_iy \ldots x_n \quad \text{(since } U \text{ satisfies eq. (5))} \\
&= f(x_1, x_2, \ldots, x_{j-1}xx_j, \ldots, x_iy, \ldots, x_n) \\
&\qquad\qquad \text{(since } U \text{ satisfies equation (5))} \\
&= h_1(\bar{x})(x_iy)(xx_j)h_2(\bar{x}) \quad \text{(from equation (7)} \\
&= f(x_1, x_2, \ldots, yxx_j, \ldots, x_n) \quad \text{(from equation (7))} \\
&= x_1x_2 \ldots x_{j-1}yxx_j \ldots x_n \quad \text{(since } U \text{ satisfies equation (5))}
\end{aligned}
$$

So U satisfies the permutation identity $x_1x_2 \ldots x_{n+2} = x_1x_2 \ldots x_{j-1}x_{j+1}x_jx_{j+2} \ldots x_{n+2}$ and hence, U is permutative. Thus the proof of the theorem is completed.

CHAPTER 5

EPIMORPHICALLY CLOSED PERMUTATIVE VARIETIES

5.1. INTRODUCTION

In this chapter, we ask the following question: **Which permutation identities are stable? or equivalently which permutation varieties are closed under epis?**

First we give some semigroup identities which are preserved under epis in conjunction with any nontrivial permutation identity and, then, we completely answer the above question and show that a permutation identity

$$x_1x_2\cdots x_n = x_{i_1}x_{i_2}\cdots x_{i_n} \quad (n \geq 3)$$

is stable iff $i_n \neq n$ $[i_1 \neq 1]$.

THEOREM 5.1.1[44, Theorem 4.7]: Let

$$x_1x_2\cdots x_n = x_{i_1}x_{i_2}\cdots x_{i_n} \tag{1}$$

be any nontrivial permutation identity. Then a nontrivial semigroup identity I(one which is not satisfied by the class of all semigroups) is preserved under epis in conjunction with (1) if I has one of the following forms:

(i) At least one side of I has no repeated variable.

(ii) $x^p = y^q$; $p, q > 0$;

(iii) $x^p = x^q$; $p, q > 0$;

(iv) $x^py^q = y^rx^s$; $p, q, r, s > 0$;

(v) $x^p = 0$; $p > 0$;

(vi) $x^py^q = 0$; $p, q > 0$;

REMARK 1: We regard $u = 0$ (for some non empty word u) as a semigroup identity. We define it to mean the conjunction of the two identities $uy = u = yu$ (in each case y is a variable not occuring in the word u).

PROOF: Take any semigroups U and S with U epimorphically embedded in S, and such that U (and hence S, by Theorem 3.1 which states that all permutation identities

are preserved under epis) satisfies the identity (1). We show that each of the identities (i) to (vi) satisfied by U is also satisfied by S.
(i) That S satisfies (i) if U does, follows from Theorem 4.1.12 as all saturated varieties of semigroups are epimorphically closed.
(ii) Assume U satisfies (ii). Then for all $u, v \in U$ we have

$$u^p = v^q = v^p = u^q.$$

Take any $x, y \in S$. We assume that $x \in S \setminus U$. By zigzag Theorem 1.2.5, we may let (1.2) be a zigzag of shortest possible length m over U with value x. Then

$$\begin{aligned}
x^p &= a_0 t_1^p \text{ (by Propostion 4.1.8 and equations (1.2))} \\
&= (y_1 a_1^2)^p t_1^p \text{ (since } y_1 a_1^2 = y_1 a_1 a_1 \ a_0 a_1 \in U \text{)} \\
&= y_1^p a_1^p (a_1 t_1)^p \text{ (by Corollary 4.1.2, since } y_1, t_1 \in S \setminus U) \\
&= y_1^p a_1^p (a_2 t_1)^p \text{ (from equations (1.2))} \\
&= y_1^p a_1^p a_2^p t_2^p \text{ (by Corollary 4.1.2, since } y_1, t_2 \in S \setminus U) \\
&= y_1^p a_1^p a_3^p t_2^p \ (a_2^p = a_3^p) \\
&= \ldots . \\
&= y_1^p a_1^p a_{2m-1}^p t_m^p \\
&= (y_1 a_1 a_{2m-1} t_m)^p \\
&= (y_1 a_1 a_{2m})^p \text{ (from equations (1.2))} \\
&= (a_0 a_{2m})^p \\
&= u^p \text{ for all } u \in U.
\end{aligned}$$

Hence $x^p = u^p$ for all $x \in S, u \in U$ and likewise $y^q = u^q$ for all $y \in S$ and $u \in U$. Therefore $x^p = u^p = u^q = y^q$ as required.

(iii) Assume U satisfies (iii) and take any $x \in S \setminus U$. we may let equations (1.2), by

Result 1.2.5, be a zigzag of shortest possible length m over U with value x. Now,

$$\begin{aligned}
x^p &= a_0^p t_1^p \text{ (by Proposition 4.1.8 and equations (1.2))}\\
&= a_0^q t_1^p \text{ (since } U \text{ satifies(iii))}\\
&= (y_1 a_1)^q t_1^p \text{ (from equations (1.2))}\\
&= y_1^q a_1^q t_1^p \text{ (by Corollary 4.1.2, since } y_1,\ t_1 \in S\backslash U)\\
&= y_1^q a_1^q t_1^P \text{ (Since } U \text{ satisfies (iii))}\\
&= y_1^q (a_1 t_1)^p \text{ (by Corollary 4.1.2, since } y_1, t_1 \in S\backslash U)\\
&= y_1^q (a_2 t_2)^p \text{ (from equations (1.2))}\\
&= y_1^q a_2^p t_2^p \text{ by Corollary 4.1.2, since } y_1,\ t_1 \in S\backslash U)\\
&= y_1^q a_2^q t_2^p \text{ (since } U \text{ satisfies (iii))}\\
&= (y_1 a_2)^q t_2^p \text{ (by Corollary 4.1.2, since } y_1,\ t_2 \in S\backslash U)\\
&= (y_1 a_3)^q t_2^p \text{ (from equations (1.2))}\\
&= \ldots\ldots\\
&= (y_m a_{2m-1})^q t_m^p\\
&= y_m{}^q a_{2m-1}{}^q t_m^p \text{ (by Corrllary 4.1.2, since } y_m, t_m \in S)\\
&= y_m{}^q a_{2m-1}{}^p t_m^p \text{ (since } U \text{ satisfies (iii))}\\
&= y_m{}^q (a_{2m-1} t_m)^p \text{ (by Corrllary 4.1.2, since } y_m, t_m \in S\backslash U)\\
&= y_m{}^q a_{2m}{}^p \text{ (from equation (1.2))}\\
&= y_m{}^q a_{2m}{}^q \text{ (since } U \text{ satisfies (iii))}\\
&= x^q \text{ (by Proposition 4.1.8 and equation (1.2)).}
\end{aligned}$$

as required.

(iv) Assume U satisfies (iv) and take any $x, y \in S$. First we consider the case where

$x \in S \backslash U$ and $y \in U$ (the case where $x \in U$ and $y \in S \backslash U$ is similar to this case). Since $x \in S \backslash U$, we may let (1.2), by Result 1.2.5, be a zigzag for x of shortest possible length m over U. Now,

$$\begin{aligned}
x^p y^q &= {y_m}^p {a_{2m}}^p y^q \quad \text{(by Proposition 4.1.8 and equation (1.2))} \\
&= {y_m}^p y^r {a_{2m}}^s \quad \text{(since } U \text{ satisfies (iv))} \\
&= {y_m}^p y^r (a_{2m-1} t_m)^s \quad \text{(from equations (1.2))} \\
&= {y_m}^p y^r {a_{2m-1}}^s {t_m}^s \quad \text{(by Corollary 4.1.2, since } y_m, t_m \in S \backslash U) \\
&= {y_m}^p {a_{2m-1}}^p y^q {t_m}^s \quad \text{(since } U \text{ satisfies (iv))} \\
&= (y_{m-1} a_{2m-1})^p y^q {t_m}^s \quad \text{(by Corollary 4.1.2, since } y_m, t_m \in S \backslash U) \\
&= (y_{m-1} a_{2m-2})^p y^q {t_m}^s \quad \text{(from equations (1.2))} \\
&= {y_{m-1}}^p {a_{2m-2}}^p y^q {t_m}^s \quad \text{(by Corollary 5.9, since } y_{m-1}, t_m \in S \backslash U) \\
&= {y_{m-1}}^p y^r {a_{2m-2}}^s {t_m}^s \quad \text{(since } U \text{ satisfies (iv))} \\
&= {y_{m-1}}^p y^r (a_{2m-2} t_m)^s \quad \text{(by Corollary 4.1.2, since } y_{m-1}, t_m \in S \backslash U) \\
&= {y_{m-1}}^p y^r (a_{2m-3} t_{m-1})^s \quad \text{(from equations (1.2))} \\
&= \ldots\ldots \\
&= {y_1}^p y^r (a_1 t_1)^s \\
&= {y_1}^p y^r {a_1}^s {t_1}^s \quad \text{(by Corollary 4.1.2, since } y_1, t_1 \in S \backslash U) \\
&= {y_1}^p {a_1}^p y^q {t_1}^s \quad \text{(since } U \text{ satisfies (iv))} \\
&= (y_1 a_1)^p y^q {t_1}^s \quad \text{(by Corollary 4.1.2, since } y_1, t_1 \in S \backslash U) \\
&= a_0^p y^q t_1^s \quad \text{(from equations (1.2))} \\
&= y^r a_0^s t_1^s \quad \text{(since } U \text{ satifies (iv))} \\
&= y^r x^s \quad \text{(by Proposition 4.1.8 and equations (1.2))}
\end{aligned}$$

as required.

So we assume next that $x, y \in S\backslash U$. By Result 1.2.5, we may let (1.2) be a zigzag for x of shortest possible length m over U then,

$$\begin{aligned}
x^p y^q &= {y_m}^p {a_{2m}}^p y^q \text{ (by Proposition 4.1.8, and equations (1.2))} \\
&= {y_m}^p y^r {a_{2m}}^s \text{ (by the first part of the proof)} \\
&= {y_m}^p y^r (a_{2m-1} t_m)^s \text{ (from equations (1.2))} \\
&= {y_m}^p y^r {a_{2m-1}}^s {t_m}^s \text{ (by Corollary 4.1.2, since } y_m, t_m \in S\backslash U) \\
&= {y_m}^p {a_{2m-1}}^p y^q {t_m}^s \text{ (by the first part of the proof)} \\
&= (y_m a_{2m-1})^p y^q {t_m}^s \text{ (by Corollary 4.1.2, since } y_m, t_m \in S\backslash U) \\
&= (y_{m-1} a_{2m-2})^p y^q {t_m}^s \text{ (from equations (1.2))} \\
&= {y_{m-1}}^p {a_{2m-2}}^p y^q {t_m}^s \text{ (by Corollary 4.1.2, since } y_{m-1}, t_m \in S\backslash U) \\
&= {y_{m-1}}^p y^r {a_{2m-2}}^s {t_m}^s \text{ (since } U \text{ satisfies (iv))} \\
&= {y_{m-1}}^p y^r (a_{2m-2} t_m)^s \text{ (by corollary 4.1.2, since } y_{m-1}, t_m \in S\backslash U) \\
&= {y_{m-1}}^p y^r (a_{2m-3} t_{m-1})^s \text{ (from equations (1.2))} \\
&= \ldots\ldots \\
&= {y_1}^p y^r (a_1 t_1)^s \\
&= {y_1}^p y^r {a_1}^s {t_1}^s \text{ (by Corollary 4.1.2, since } y_1, t_1 \in S\backslash U) \\
&= {y_1}^p {a_1}^p y^q {t_1}^s \text{ (since } U \text{ satisfies (iv))} \\
&= (y_1 a_1)^p y^q {t_1}^s \text{ (by Corollary 4.1.2, since } y_1, t_1 \in S\backslash U) \\
&= a_0^p y^q t_1^s \text{ (from equations (1.2))} \\
&= y^r a_0^s t_1^s \text{ (since } U \text{ satifies (iv))} \\
&= y^r x^s \text{ (by Proposition 4.1.8, and equations (1.2))}
\end{aligned}$$

as required. This completes the proof of the part (iv).

(v) Assume U satisfies $x^p = 0$ and take $x, y \in S$; we show that

$$x^p y = y x^p = x^p.$$

CASE(a): $x \in S \backslash U$, $y \in U$, by Result 1.2.5, let (1.2), be a zigzag for x over U of shortest possible length m. Then,

$$\begin{aligned} x^p y &= y_m^p a_{2m}{}^p y \text{ (by Proposition 4.1.8, and equations (1.2))} \\ &= y_m^p a_{2m}{}^p \text{ (since } U \text{ satisfies (v))} \\ &= x^p. \end{aligned}$$

Similarly $yx^p = x^p$, as required.

CASE(b): $x \in U, y \in s\backslash U$. let (1.2), by Result 1.2.5, be a zigzag of length m over U with value y. Then,

$$\begin{aligned} x^p y &= x^p a_0 t_1 = x^p a_1 t_1 \text{ (from equations (1.2) and since } u \text{ satifies (v))} \\ &= x^p a_2 t_2 \text{ (from equations (1.2))} \\ &= x^p a_3 t_2 \text{ (since } U \text{ satifies(v))} \\ &= \ldots\ldots \\ &= x^p a_{2m-1} t_m \\ &= x^p a_{2m} \\ &= x^p. \text{ (from equations (1.2) and since } U \text{ satisfies (v))} \end{aligned}$$

Similarly $yx^p = x^p$, as required.

CASE(c): $x, y \in S\backslash U$. By Result 1.2.5, we may let (1.2), be a zigzag for x of shortest possible length m over U. Then,

$$\begin{aligned} x^p y &= y_m{}^p a_{2m}{}^p y \text{ (by Proposition 4.1.8, and equations (1.2))} \\ &= y_m{}^p a_{2m}{}^p \text{ (from case (b) above)} \\ &= x^p. \end{aligned}$$

Similarly $yx^p = x^p$, as required.

(vi) Assume U satisfies (vi) and take any $x, y, z \in S$; we prove that

$$x^p y^q z \;=\; z x^p y^q \;=\; x^p y^q.$$

CASE(a): $x, y, z \in U, z \in S \backslash U$. By Result 1.2.5, we may let (1.2) be a zigzag of shortest possible length m over U with value z. Then,

$$\begin{aligned}
x^p y^q z &= x^p y^q a_0 t_1 \quad \text{(from equations (1.2))}\\
&= x^p y^q a_1 t_1 \quad \text{(since } U \text{ satisfies (vi))}\\
&= x^p y^q a_2 t_2 \quad \text{(from equations (1.2))}\\
&= \ldots\ldots\\
&= x^p y^q a_{2m-2} t_m\\
&= x^p y^q a_{2m-1} t_m \quad \text{(since } U \text{ satisfies (vi))}\\
&= x^p y^q a_{2m} \quad \text{(from equations (1.2))}\\
&= x^p y^q \quad \text{(since } U \text{ satisfies (vi))}
\end{aligned}$$

By a similar argument we can show easily that $z x^p y^q = x^p y^q$. Therefore $x^p y^q z = z x^p y^q = x^p y^q$, as required.

CASE(b): $y, z \in U, x \in S \backslash U$. As $x \in S \backslash U$, by Result 1.2.5, we may let (1.2) be a zigzag of shortest possible length m over U with value x. Then,

$$\begin{aligned}
x^p y^q z &= {y_m}^p {a_{2m}}^p y^q x \quad \text{(by Proposition 4.1.8, and equations (1.2))}\\
&= {y_m}^p {a_{2m}}^p y^q \quad \text{(since } U \text{ satisfies (vi))}\\
&= x^p y^q \quad \text{(by Proposition 4.1.8 and equations (1.2))}
\end{aligned}$$

Also

$$\begin{aligned}
z x^p y^q &= z {y_m}^p {a_{2m}}^p y^q \quad \text{(by Proposition 4.1.8, and equations (1.2))}\\
&= {y_m}^p {a_{2m}}^p y^q \quad \text{(from case (a))}\\
&= x^p y^q \quad \text{(by Proposition 4.1.8, and equations (1.2))}
\end{aligned}$$

Therefore $x^p y^q z = z x^p y^q = x^p y^q$, as required.

CASE(c): $x, z \in U, y \in S\backslash U$. This case is dual to case (b).

CASE(d): $z \in U, x, y \in S\backslash U$. By Result 1.2.5, we may let (1.2) be a zigzag of shortest possible length m over U with value x. Now,

$$\begin{aligned} x^p y^q z &= {y_m}^p {a_{2m}}^p y^q z \text{ (by Proposition 4.1.8, and equations (1.2))} \\ &= {y_m}^p {a_{2m}}^p y^q \text{ (from case (c))} \\ &= x^p y^q \text{ (by Proposition 4.1.8, and equations (1.2))} \end{aligned}$$

Since $y \in S\backslash U$, by Result 1.2.5, we may let $y = b_0 z_1 = s_1 b_1 z_1$ be the first two lines of zigzag for y with $b_0, b_1 \in U$ and $s_1, z_1 \in S\backslash U$. Now

$$\begin{aligned} z x^p y^q &= z {y_m}^p {a_{2m}}^p {b_0}^q {z_1}^q \text{ (by Proposition 4.1.8)} \\ &= {y_m}^p {a_{2m}}^p {b_0}^q {z_1}^q \text{ (from case (a))} \\ &= x^p y^q \text{ (by Proposition 4.1.8)} \end{aligned}$$

Therefore $x^p y^q z = z x^p y^q = x^p y^q$, as required.

CASE(e): $y \in U, x, z \in S\backslash U$ or $x \in U$, $y, z \in S\backslash U$ or $x, y, z \in S\backslash U$. As $z \in S\backslash U$, by Result 1.2.5, we may let (1.2) be a zigzag of shortest possible length m over U with value z. Now,

$$\begin{aligned} x^p y^q z &= x^p y^q a_0 t_1 \text{ (from equations (1.2))} \\ &= x^p y^q a_1 t_1 \text{ (from cases (b), (c) and (d))} \\ &= x^p y^q a_2 t_2 \text{ (from equations (1.2))} \\ &= \ldots\ldots \\ &= x^p y^q a_{2m-2} t_m \\ &= x^p y^q a_{2m-1} t_m \text{ (from cases (b), (c) and (d))} \\ &= x^p y^q a_{2m} \text{ (from equations (1.2))} \\ &= x^p y^q \text{ (from case (d)).} \end{aligned}$$

The dual argument shows that $zx^py^q = x^py^q$.
Therefore $x^py^qz = zx^py^q = x^py^q$, as required, thus completing the proof of the Theorem 5.1.1.

The following theorem is from Khan [48, 49] and extends Theorems 5.1.1(ii) and 5.1.1(iii).

THEOREM 5.1.2: Let (1) be any nontrivial permutation identity. Then any nontrivial homotypical identity I (one which is not satisfied by the class of all semigroups) of the following forms are preserved under epis in conjunction with (1):

(i) $x_1^{p_1}x_2^{p_2}...x_r^{p_r} = x_1^{q_1}x_2^{q_2}...x_r^{q_r}$, where $0 < p_r \leq p_{r-1} \leq ... \leq p_2 \leq p_1$ and $0 < q_1 \leq q_2 \leq ... \leq q_{r-1} \leq q_r$ and $r \geq 0$;

(ii) $x_1^px_2^p...x_r^p = y_1^qy_2^q...y_r^q$, for any $p, q > 0$.

PROOF: Take any semigroups U and S with U epimorphically embedded in S, and such that U (and, hence, S, by Theorem 3.1) satisfies the identity (1). We show that each of the identities (i) and (ii) satisfied by U is also satisfied by S.

(i). Assume that U satisfies (i). For $k = 1, 2, ..., r$; consider the word $x_1^{p_1}x_2^{p_2}...x_k^{p_k}$ of length $p_1 + p_2 + ... + p_k$. We shall prove that S satisfies (i) by induction on k, assuming that the remaining elements $x_{k+1}, x_{k+2}, ..., x_r \in U$.
First for $k = 0$, the equation (i) is satisfied by S vacuously. So assume next that (i) is satisfied for all $x_1, x_2, ..., x_{k-1} \in S$ and all $x_k, x_{k+1}, ..., x_r \in U$. Without loss we can assume that $x_k \in S\backslash U$. As $x_k \in S\backslash U$ and $Dom(U, S) = S$, by Result 1.2.5, we may let (1.2) be a zigzag of shortest possible length m over U with value x_k. We assume first that $1 < k < r$. Now

$$\begin{aligned} x_1^{p_1}x_2^{p_2}...x_r^{p_r} &= x_1^{p_1}x_2^{p_2}...a_0^{p_k}t_1^{p_k}x_{k+1}^{p_{k+1}}...x_r^{p_r} \quad \text{(by eq. (1.2) and Result 2.2.5)} \\ &= x_1^{p_1}x_2^{p_2}...a_0^{p_k}b_{k+1}^{(1)p_k}b_{k+2}^{(1)p_k}...b_r^{(1)p_k}t_1^{(1)p_k}x_{k+1}^{p_{k+1}}...x_r^{p_r} \\ &\quad \text{(by eq. (1) and Results 2.2.3 and 2.2.4, for some } b_{k+1}^{(1)}, b_{k+2}^{(1)}, ... \\ &\quad b_r^{(1)} \in U \text{ and } t_1^{(1)} \in S\backslash U) \\ &= x_1^{p_1}x_2^{p_2}...a_0^{p_k}b_{k+1}^{(1)p_{k+1}} \\ &\quad \text{(where } w^{(1)} = b_{k+1}^{(1)p_k-p_{k+1}}b_{k+2}^{(1)p_k-p_{k+2}}...b_r^{(1)p_k-p_r} \\ &\quad \text{and } z = x_{k+1}^{p_{k+1}}...x_r^{p_r}, \text{by Result 2.2.4)} \end{aligned}$$

$= \quad x_1{}^{q_1}x_2{}^{q_2}...x_{k-1}{}^{q_{k-1}}a_0{}^{q_k}b_{k+1}{}^{(1)^{q_{k+1}}}b_{k+2}{}^{(1)^{q_{k+2}}}...b_r{}^{(1)^{q_r}}w^{(1)}t_1{}^{(1)^{p_k}}z$
(by inductive hypothesis as $a_0 \in U$)

$= \quad vy_1{}^{(1)^{q_k}}c_1{}^{(1)^{q_k}}c_2{}^{(1)^{q_k}}...c_{k-1}{}^{(1)^{q_k}}a_1{}^{q_k}b_{k+1}{}^{(1)^{q_{k+1}}}b_{k+2}{}^{(1)^{q_{k+2}}}...b_r{}^{(1)^{q_r}}w^{(1)}$
$t_1{}^{(1)^{p_k}}z$ (by eq. (1) and Result 2.2.5 and dual of Result 2.2.4, for some $c_1{}^{(1)}, c_2{}^{(1)}, ..., c_{k-1}{}^{(1)} \in U$, $v = x_1{}^{q_1}x_2{}^{q_2}...x_{k-1}{}^{q_{k-1}}$ and $y_1{}^{(1)}, t_1{}^{(1)} \in S\backslash U$)

$= \quad vy_1{}^{(1)^{q_k}}v^{(1)}c_1{}^{(1)^{q_1}}c_2{}^{(1)^{q_2}}...c_{k-1}{}^{(1)^{q_{k-1}}}a_1{}^{q_k}b_{k+1}{}^{(1)^{q_{k+1}}}b_{k+2}{}^{(1)^{q_{k+2}}}...b_r{}^{(1)^{q_r}}w^{(1)}$
$t_1{}^{(1)^{p_k}}z$ (by Result 2.2.5, where $v^{(1)} = c_1{}^{(1)^{q_k-q_1}}c_2{}^{(1)^{q_k-q_2}}...c_{k-1}{}^{(1)^{q_k-q_{k-1}}}$
as $y_1{}^{(1)}, t_1{}^{(1)} \in S\backslash U$)

$= \quad vy_1{}^{(1)^{q_k}}v^{(1)}c_1{}^{(1)^{p_1}}c_2{}^{(1)^{p_2}}...c_{k-1}{}^{(1)^{p_{k-1}}}a_1{}^{p_k}b_{k+1}{}^{(1)^{p_{k+1}}}b_{k+2}{}^{(1)^{p_{k+2}}}...b_r{}^{(1)^{p_r}}w^{(1)}$
$t_1{}^{(1)^{p_k}}z$ (*as* U satisfies (i))

$= \quad vy_1{}^{(1)^{q_k}}v^{(1)}c_1{}^{(1)^{p_1}}c_2{}^{(1)^{p_2}}...c_{k-1}{}^{(1)^{p_{k-1}}}a_1{}^{p_k}b_{k+1}{}^{(1)^{p_k}}b_{k+2}{}^{(1)^{p_k}}...b_r{}^{(1)^{p_k}}$
$t_1{}^{(1)^{p_k}}z$ (by Result 2.2.5, as $y_1{}^{(1)}, y_1{}^{(1)} \in S\backslash U$ and
$w^{(1)} = b_{k+1}{}^{(1)^{p_k-p_{k+1}}}b_{k+2}{}^{(1)^{p_k-p_{k+2}}}...b_r{}^{(1)^{p_k-p_r}}$)

$= \quad vy_1{}^{(1)^{q_k}}v^{(1)}c_1{}^{(1)^{p_1}}c_2{}^{(1)^{p_2}}...c_{k-1}{}^{(1)^{p_{k-1}}}a_1{}^{p_k}t_1{}^{p_k}z$
(by Result 2.2.5, as $t_1{}^{p_k} = b_{k+1}{}^{(1)^{p_k}}b_{k+2}{}^{(1)^{p_k}}...b_r{}^{(1)^{p_k}}t_1{}^{(1)^{p_k}}$)

$= \quad vy_1{}^{(1)^{q_k}}v^{(1)}c_1{}^{(1)^{p_1}}c_2{}^{(1)^{p_2}}...c_{k-1}{}^{(1)^{p_{k-1}}}a_2{}^{p_k}t_2{}^{p_k}z$
(by Result 2.2.5 and equations (1))

$= \quad ......$

$= \quad vy_{m-1}{}^{(1)^{q_k}}v^{(m-1)}c_1{}^{(m-1)^{p_1}}c_2{}^{(m-1)^{p_2}}...c_{k-1}{}^{(m-1)^{p_{k-1}}}a_{2m-2}{}^{p_k}t_m{}^{p_k}z$

$= \quad vy_{m-1}{}^{(1)^{q_k}}v^{(m-1)}c_1{}^{(m-1)^{p_1}}c_2{}^{(m-1)^{p_2}}...c_{k-1}{}^{(m-1)^{p_{k-1}}}a_{2m-2}{}^{p_k}b_{k+1}{}^{(m)^{p_k}}$
$b_{k+2}{}^{(m)^{p_k}}...b_r{}^{(m)^{p_k}}t_m{}^{(1)^{p_k}}z$ (by Results 2.2.4 and 2.2.5
as $y_{m-1}{}^{(1)}, t_m{}^{(1)} \in S\backslash U$ for some $b_{k+1}{}^{(m)}, b_{k+2}{}^{(m)}..., b_r{}^{(m)} \in U$)

$= \quad vy_{m-1}{}^{(1)^{q_k}}v^{(m-1)}c_1{}^{(m-1)^{p_1}}c_2{}^{(m-1)^{p_2}}...c_{k-1}{}^{(m-1)^{p_{k-1}}}a_{2m-2}{}^{p_k}b_{k+1}{}^{(m)^{p_{k+1}}}$
$b_{k+2}{}^{(m)^{p_{k+2}}}...b_r{}^{(m)^{p_r}}w^{(m)}t_m{}^{(1)^{p_k}}z$ (by Result 2.2.5 as $y_{m-1}{}^{(1)}, t_m{}^{(1)} \in S\backslash U$,
where $w^{(m)} = b_{k+1}{}^{(m)^{p_k-p_{k+1}}}b_{k+2}{}^{(m)^{p_k-p_{k+2}}}...b_r{}^{(m)^{p_k-p_r}}$)

$= \quad vy_{m-1}{}^{(1)^{q_k}}v^{(m-1)}c_1{}^{(m-1)^{q_1}}c_2{}^{(m-1)^{q_2}}...c_{k-1}{}^{(m-1)^{q_{k-1}}}a_{2m-2}{}^{q_k}b_{k+1}{}^{(m)^{q_{k+1}}}$
$b_{k+2}{}^{(m)^{q_{k+2}}}...b_r{}^{(m)^{q_r}}w^{(m)}t_m{}^{(1)^{p_k}}z$ (since U satisfies (i))

$= \quad vy_{m-1}{}^{(1)^{q_k}}c_1{}^{(m-1)^{q_k}}c_2{}^{(m-1)^{q_k}}...c_{k-1}{}^{(m-1)^{q_k}}a_{2m-2}{}^{q_k}b_{k+1}{}^{(m)^{q_{k+1}}}$
$b_{k+2}{}^{(m)^{q_{k+2}}}...b_r{}^{(m)^{q_r}}w^{(m)}t_m{}^{(1)^{p_k}}z$
(by Result 2.2.4, as $v^{(m-1)} = c_1{}^{(m-1)^{q_k-q_1}}c_2{}^{(m-1)^{q_k-q_2}}...c_{k-1}{}^{(m-1)^{q_k-q_{k-1}}}$
and $y_{m-1}{}^{(1)}, t_m{}^{(1)} \in S\backslash U$)

$= \quad vy_{m-1}{}^{q_k}a_{2m-2}{}^{q_k}b_{k+1}{}^{(m)^{q_{k+1}}}b_{k+2}{}^{(m)^{q_{k+2}}}...b_r{}^{(m)^{q_r}}w^{(m)}t_m{}^{(1)^{p_k}}z$
(by Result 2.2.4, as $y_{(m-1)}{}^{q_k} = y_{m-1}{}^{(1)^{q_k}}c_1{}^{(m-1)^{q_k}}$
$c_2{}^{(m-1)^{q_k}}...c_{k-1}{}^{(m-1)^{q_k}}$ and $y_{m-1}{}^{(1)}, t_m{}^{(1)} \in S\backslash U$)

$= \quad vy_m{}^{q_k}a_{2m-1}{}^{q_k}b_{k+1}{}^{(m)^{q_{k+1}}}b_{k+2}{}^{(m)^{q_{k+2}}}...b_r{}^{(m)^{q_r}}w^{(m)}t_m{}^{(1)^{p_k}}z$
(by Result 2.2.4 and equations (1) as $y_{m-1}{}^{(1)}, t_m{}^{(1)} \in S\backslash U$)

$= \quad vy_m{}^{(1)^{q_k}}c_1{}^{(m)^{q_k}}c_2{}^{(m)^{q_k}}...c_{k-1}{}^{(m)^{q_k}}a_{2m-1}{}^{q_k}b_{k+1}{}^{(m)^{q_{k+1}}}$
$b_{k+2}{}^{(m)^{q_{k+2}}}...b_r{}^{(m)^{q_r}}w^{(m)}t_m{}^{(1)^{p_k}}z$
(by Result 2.2.4 and dual of Result 2.2.3 for some
$c_1{}^{(m)}, c_2{}^{(m)}, ...c_{k-1}{}^{(m)} \in U$ and $y_m{}^{(1)} \in S\backslash U$)

$= \quad vy_m{}^{(1)^{q_k}}v^{(m)}c_1{}^{(m)^{q_1}}c_2{}^{(m)^{q_2}}...c_{k-1}{}^{(m)^{q_{k-1}}}a_{2m-1}{}^{q_k}b_{k+1}{}^{(m)^{q_{k+1}}}$
$b_{k+2}{}^{(m)^{q_{k+2}}}...b_r{}^{(m)^{q_r}}w^{(m)}t_m{}^{(1)^{p_k}}z$
(by Result 2.2.4 as $y_m{}^{(1)}, t_m{}^{(1)} \in S\backslash U$,
where $v^{(m)} = c_1{}^{(m)^{q_k-q_1}}c_2{}^{(m)^{q_k-q_2}}...c_{k-1}{}^{(m)^{q_k-q_{k-1}}}$)

$= \quad vy_m{}^{(1)^{q_k}}v^{(m)}c_1{}^{(m)^{p_1}}c_2{}^{(m)^{p_2}}...c_{k-1}{}^{(m)^{p_{k-1}}}a_{2m-1}{}^{p_k}b_{k+1}{}^{(m)^{p_{k+1}}}b_{k+2}{}^{(m)^{p_{k+2}}}...b_r{}^{(m)^{p_r}}$
$w^{(m)}t_m{}^{(1)^{p_k}}z$ (as U satisfies (i))

$= \quad vy_m{}^{(1)^{q_k}}v^{(m)}c_1{}^{(m)^{p_1}}c_2{}^{(m)^{p_2}}...c_{k-1}{}^{(m)^{p_{k-1}}}a_{2m-1}{}^{p_k}b_{k+1}{}^{(m)^{p_k}}b_{k+2}{}^{(m)^{p_k}}...b_r{}^{(m)^{p_k}}$
$t_m{}^{(1)^{p_k}}z$ (by Result 2.2.4, as $y_m{}^{(1)}, t_m{}^{(1)} \in S\backslash U$ and
$w^{(m)} = b_{k+1}{}^{(m)^{p_k-p_{k+1}}}b_{k+2}{}^{(m)^{p_k-p_{k+2}}}...b_r{}^{(m)^{p_k-p_r}}$)

$= \quad vy_m{}^{(1)^{q_k}}v^{(m)}c_1{}^{(m)^{p_1}}c_2{}^{(m)^{p_2}}...c_{k-1}{}^{(m)^{p_{k-1}}}a_{2m-1}{}^{p_k}t_m{}^{p_k}z$
(by Result 2.2.4, as $y_m{}^{(1)}, t_m \in S\backslash U$ and
$b_{k+1}{}^{(m)^{p_k}}b_{k+2}{}^{(m)^{p_k}}...b_r{}^{(m)^{p_k}}t_m{}^{(1)^{p_k}} = t_m{}^{p_k}$)

$= \quad vy_m{}^{(1)^{q_k}}v^{(m)}c_1{}^{(m)^{p_1}}c_2{}^{(m)^{p_2}}...c_{k-1}{}^{(m)^{p_{k-1}}}a_{2m}{}^{p_k}z$
(by Result 2.2.4 and equations (1), as $y_m{}^{(1)}, t_m \in S\backslash U$)

$= \quad vy_m{}^{(1)^{q_k}}v^{(m)}c_1{}^{(m)^{p_1}}c_2{}^{(m)^{p_2}}...c_{k-1}{}^{(m)^{p_{k-1}}}a_{2m}{}^{p_k}x_{k+1}{}^{p_{k+1}}...x_r{}^{p_r}$
$(as\ z = x_{k+1}{}^{p_{k+1}}...x_r{}^{p_r})$

$$= \; vy_m{}^{(1)^{q_k}} v^{(m)} c_1{}^{(m)^{q_1}} c_2{}^{(m)^{q_2}} ... c_{k-1}{}^{(m)^{q_{k-1}}} a_{2m}{}^{q_k} x_{k+1}{}^{q_{k+1}} ... x_r{}^{q_r}$$

(as U satisfies (i))

$$= \; vy_m{}^{(1)^{q_k}} c_1{}^{(m)^{q_k}} c_2{}^{(m)^{q_k}} ... c_{k-1}{}^{(m)^{q_k}} a_{2m}{}^{q_k} x_{k+1}{}^{q_{k+1}} ... x_r{}^{q_r}$$

(by Result 2.2.4, as $y_m{}^{(1)}, t_m \in S \backslash U$ and $v^m \; = \; c_1{}^{(m)^{q_k - q_1}} c_2{}^{(m)^{q_k - q_2}} ... c_{k-1}{}^{(m)^{q_k - q_{k-1}}}$)

$$= \; vy_m{}^{q_k} a_{2m}{}^{q_k} x_{k+1}{}^{q_{k+1}} ... x_r{}^{q_r}$$

(by Result 2.2.4, as $y_m{}^{q_k} \; = \; y_m{}^{(1)^{q_k}} c_1{}^{(m)^{q_k}} c_2{}^{(m)^{q_k}} ... c_{k-1}{}^{(m)^{q_k}}$)

$$= \; x_1{}^{q_1} x_2{}^{q_2} ... x_r{}^{q_r} \text{ (By Result 2.2.5 as } v \; = \; x_1{}^{q_1} x_2{}^{q_2} ... x_{k-1}{}^{q_{k-1}}\text{)},$$

as required.

Finally, a proof in the remaining cases, namely when $k \; = \; 1 \; or \; k \; = \; r$, can be obtained from the proof above by making the following conventions:

First when k = 1,

(i) the word v = 1,

(ii) the word

$$c_1{}^{(i)^{q_k}} c_2{}^{(i)^{q_k}} ... c_{k-1}{}^{(i)^{q_k}} \; = \; c_1{}^{(i)^{q_1}} c_2{}^{(i)^{q_2}} ... c_{k-1}{}^{(i)^{q_{k-1}}} \; = \; 1 \quad for \; i \; = \; 1, 2, ..., m \quad \text{and}$$

$$c_1{}^{(i)^{p_1}} c_2{}^{(i)^{p_2}} ... c_{k-1}{}^{(i)^{p_{k-1}}} \; = \; v^{(i)} \; = 1 \; and \; y_i{}^{(1)} \; = \; y_i \quad for \; i \; = \; 1, 2, ..., m.$$

Dually when k = r,

(i) word z = 1,

(ii) the word

$$b_{k+1}{}^{(i)^{p_k}} b_{k+2}{}^{(i)^{p_k}} ... b_r{}^{(i)^{p_k}} \; = \; b_{k+1}{}^{(i)^{p_{k+1}}} b_{k+2}{}^{(i)^{p_{k+2}}} ... b_r{}^{(i)^{p_r}} \; = \; 1 \quad for \; i \; = \; 1, 2, ..., m.$$

$$\text{and} \qquad b_{k+1}{}^{(i)^{q_{k+1}}} b_{k+2}{}^{(i)^{q_{k+2}}} ... b_r{}^{(i)^{q_r}} \; = \; w^{(i)} \; = \; 1 \quad for \; i \; = \; 1, 2, ..., m.$$

(ii). To prove (ii), we first note that for all $u_1, u_2, ..., u_r, v_1, v_2, ..., v_r \in U$,

$$u_1{}^p u_2{}^p ... u_r{}^p \; = \; v_1{}^q v_2{}^q ... v_r{}^q \; = \; v_1{}^p v_2{}^p ... v_r{}^p \; = \; u_1{}^q u_2{}^q ... u_r{}^q. \tag{3}$$

Now first we prove that for all $x_1, x_2, ..., x_r \in S$ and $u_1, u_2, ..., u_r \in U$,

$$x_1{}^p x_2{}^p ... x_r{}^p \; = \; u_1{}^p u_2{}^p ... u_r{}^p. \tag{4}$$

Assume that U satisfies (4). For k = 1,2,...,r; consider the word $x_1{}^p x_2{}^p ... x_k{}^p$ of length kp. We shall prove that S satisfies (4) by induction on k , assuming that the remaining elements $x_{k+1}, x_{k+2}, ..., x_r \in U$.

First for $k = 0$, the equation (4) is satisfied by S vacuously. So assume next that (4) is true for all $x_1, x_2, ..., x_{k-1} \in S$ and all $x_k, x_{k+1}, ..., x_r \in U$. Without loss we can assume that $x_k \in S \backslash U$. As $x_k \in S\backslash U$ and $Dom(U,S) = S$, by Result 2.1, we may let (1) be a zigzag of shortest possible length m over U with value x_k. We assume first that $1 \leq k < r$.

Now

$$x_1{}^p x_2{}^p ... x_{k-1}{}^p a_0{}^p t_1{}^p x_{k+1}{}^p ... x_r{}^p =$$

$= x_1{}^p x_2{}^p ... x_{k-1}{}^p a_0{}^p b_{k+1}{}^{(1)^p} b_{k+2}{}^{(1)^p} ... b_r{}^{(1)^p} t_1{}^{(1)^p} z$
(by Results 2.2.1, 2.2.3 and 2.2.4 for some $t_1{}^{(1)} \in S\backslash U$ and $b_{k+1}{}^{(1)}, b_{k+2}{}^{(1)}, ..., b_r{}^{(1)} \in U$) and $z = x_{k+1}{}^p ... x_r{}^p$

$= x_1{}^p x_2{}^p ... x_{k-1}{}^p (y_1 a_1{}^2)^p b_{k+1}{}^{(1)^p} b_{k+2}{}^{(1)^p} ... b_r{}^{(1)^p} t_1{}^{(1)^p} z$
(by inductive hypothesis, as $y_1 a_1{}^2 = y_1 a_1 a_1$
$= a_0 a_1 \in U$)

$= x_1{}^p x_2{}^p ... x_{k-1}{}^p (y_1 a_1)^p a_1{}^p b_{k+1}{}^{(1)^p} b_{k+2}{}^{(1)^p} ... b_r{}^{(1)^p} t_1{}^{(1)^p} z$
(by Result 2.2.4, as $y_1, t_1{}^{(1)} \in S\backslash U$)

$= x_1{}^p x_2{}^p ... x_{k-1}{}^p (y_1 a_1)^p a_1{}^p t_1{}^p z$ (by Result 2.2.4, as $b_{k+1}{}^{(1)^p} b_{k+2}{}^{(1)^p} ... b_r{}^{(1)^p} t_1{}^{(1)^p} = t_1{}^p$)

$= x_1{}^p x_2{}^p ... x_{k-1}{}^p (y_1 a_1)^p a_2{}^p t_2{}^p z$ (by Result 2.2.4, as $y_1, t_2 \in S\backslash U$)

$= x_1{}^p x_2{}^p ... x_{k-1}{}^p (y_1 a_1)^p a_3{}^p t_2{}^p z$ (by inductive hypothesis, $as\ y_1 a_1 = a_0$ and $a_0,\ a_2 \in U$)

$= x_1{}^p x_2{}^p ... x_{k-1}{}^p (y_1 a_1)^p a_4{}^p t_3{}^p z$ (by equations (1) and Result 2.2.4)

$= x_1{}^p x_2{}^p ... x_{k-1}{}^p (y_1 a_1)^p a_5{}^p t_3{}^p z$ (by inductive hypothesis, as $y_1 a_1 = a_0$ and $a_0,\ a_4 \in U$)

$= ......$

$= x_1{}^p x_2{}^p ... x_{k-1}{}^p (y_1 a_1)^p a_{2m-1}{}^p t_m{}^p z$

$= \; x_1{}^p x_2{}^p \ldots x_{k-1}{}^p a_0{}^p a_{2m}{}^p x_{k+1}{}^p \ldots x_r{}^p$ (by equations (1) and Result 2.2.4 $as\ z \;=\; x_{k+1}{}^p \ldots x_r{}^p$ and $y_1 a_1 \;=\; a_0$)

$= \; x_1{}^p x_2{}^p \ldots x_{k-1}{}^p (a_0 a_{2m})^p x_{k+1}{}^p \ldots x_r{}^p$ (by Result 2.2.4 since $a_0 \;=\; y_1 a_1$, $a_{2m} \;=\; a_{2m-1} t_m \ and\ y_1, t_m \in S \backslash U$)

$= \; u_1{}^p u_2{}^p \ldots u_{k-1}{}^p u_k{}^p u_{k+1}{}^p \ldots u_r{}^p$ (by inductive hypothesis as $a_0 a_{2m}$, $u_1, u_2, \ldots, u_r \in U$).

Therefore, by induction

$$x_1{}^p x_2{}^p \ldots x_r{}^p \;=\; u_1{}^p u_2{}^p \ldots u_r{}^p$$

for all $x_1, x_2, \ldots, x_r \in S \ and\ u_1, u_2, \ldots, u_r \in U$.

Finally, a proof in the remaining case, namely when $k = r$, can be obtained from the proof above by making the following conventions:

(i) word z = 1,

(ii) the word

$$b_{k+1}{}^{(1)}, b_{k+2}{}^{(1)}, \ldots, b_r{}^{(1)} \;=\; z \;=\; 1 \text{ and } t_1{}^{(1)} \;=\; t_1.$$

This completes the proof of (4).
Similarly

$$y_1{}^q y_2{}^q \ldots y_r{}^q \;=\; u_1{}^q u_2{}^q \ldots u_r{}^q$$

for all $y_1, y_2, \ldots, y_r \in S$ and $u_1, u_2, \ldots, u_r \in U$.

Now

$$x_1{}^p x_2{}^p \ldots x_r{}^p \;=\; u_1{}^p u_2{}^p \ldots u_r{}^p \;=\; u_1{}^q u_2{}^q \ldots u_r{}^q \;=\; y_1{}^q y_2{}^q \ldots y_r{}^q \quad \text{(by equations (3))},$$

as required.

5.2. EPIMORPHICALLY STABLE PERMUTATION IDENTITIES

In this section, we determine all permutation identities which are stable and show that any permutation identity is stable if and only if $i_n \neq n [i_1 \neq 1]$

In the following theorem, bracketed statements are dual to the other statements. Contents of this section are from [44].

THEOREM 5.2.1: Let $x_1x_2x_3.....x_n = x_{i_1}x_{i_2}....x_{i_n}$ (1)
be any permutation identity with $n \geq 3$ and such that $i_n \neq n[i_1 \neq 1]$. Then all identities, in conjunction with (1), are preserved under epis.

PROOF: Take any identity

$$u(x_1, x_2, x_3....., x_p) = v(x_1, x_2, x_3....., x_p) \tag{2}$$

and any semigroup U and S such that U is a subsemigroup of S and U satisfies (1) and (2), and $Dom(U, S) = S$.

By the Theorem 3.1, S satisfies (1). Now we show that S satisfies (2). Since S satisfies (1), S also satisfies the permutation identity

$$x_1x_2x_3......x_nxy = x_1x_2x_3......x_nyx. \tag{3}$$

LEMMA 5.2.2: Take any word w in variables $x_1, x_2, x_3....., x_k$ and $a_1, a_2, a_3....., a_n \in U$ and any $t_1, t_2, t_3....., t_k \in S^1$ such that if $t_i \in S$ then
$a_i = y_ib_i$ for some $y_i \in S\backslash U$, $b_i \in S$(i = 1,2...,k). Then

$$w(a_1t_1, a_2t_2, ..., a_kt_k) = w(a_1, a_2, a_3....., a_k)w(t_1, t_2, t_3....., t_k)$$

PROOF: Let x_q be the first variable appearing in w for which $tq \in S$ (whence $a_q = y_qb_q$ for some $y_q \in S\backslash U, b_q \in S$). Then

$$\begin{aligned} w(a_1t_1, a_2t_2, ..., a_kt_k) &= w(a_1t_1, a_2t_2, ..., y_qb_qt_q, ..., a_kt_k) \\ &= w(a_1, a_2, a_3....., y_qb_q, ...a_k)w(t_1, t_2, t_3....., t_k) \\ &\quad \text{(by Corollary (4.1.4))} \\ &= w(a_1, a_2, a_3....., a_k)w(t_1, t_2, t_3....., t_k) \end{aligned}$$

as required.

LEMMA 5.2.3: Let U be any subsemigroup of any semigroup S. Take any d_1 , $d_2 \in S$ such that their exist zigzags over $U \in S$ with values d_1 and d_2. Then their exist zigzags for d_1 and d_2 of some common length over U.

PROOF: It is sufficient to prove that given any zigzag for any element $d \in S$ of length m, we obtain a zigzag for d of length $m+1$. Let (1.2) be a zigzag of length m for d over U. Then $d = a_0t_1 = y_1a_1t_1 = y_1a_1t_1 = y_1a_1t_1 = \cdots = y_ma_{2m}$, is essentially a zigzag of length $m+1$ over U with value d as required.

From Lemma 5.2.2, $d_1, d_2, \cdots, d_n$ all have zigzags over U in S^1 of the same length, say

$$
\begin{aligned}
d_i &= a_0^{(i)} t_1^{(i)}, \quad a_0^{(i)} = y_1^{(i)} a_1^{(i)} \\
y_k^{(i)} a_{2k}^{(i)} &= y_{k+1}^{(i)} a_{2k+1}^{(i)}, \quad a_{2k-1}^{(i)} t_k^{(i)} = a_{2k}^{(i)} t_{k+1}^{(i)} \qquad (4) \\
a_{2m-1}^{(i)} t_m^{(i)} &= a_{2m}^{(i)} y_m^{(i)}, \quad a_{2m}^{(i)} = d_i (\, i = 1, 2, \ldots, n, \ k = 1, 2, \ldots, m-1),
\end{aligned}
$$

where $a_j^{(i)} \in U(\, i = 1, 2, \ldots, n, \ j = 0, 1, \ldots, 2m)$ and $t_q^{(i)}, \ y_q^{(i)} \in S^1$ $(i = 1, 2, \ldots, n,$ $q = 1, 2, \ldots, m)$.

we denote $S \times S \times \cdots \times S$, the Cartesian product of n copies of S, by S^n.

We return to the proof of Theorem 5.2.1. Take any $d_1, d_2, d_3 ....., d_p \in S$. If some $d_i \in U$, there is a zigzag in S^1 over U with value d_i, namely, $d_i = d_i 1 = 1 d_i 1 = 1 d_i$. Now $d_1, d_2, d_3 ....., d_p$ all have zigzag over U in S^1 of some common length, by Lemma 5.2.3, say

$$d_i = a_0{}^{(i)} t_1{}^{(i)}, a_0{}^{(i)} = y_1{}^{(i)} a_1{}^{(i)}$$

$$y_k{}^{(i)} a_{2k}{}^{(i)} = y_{k+1}{}^{(i)}, a_{2k+1}{}^{(i)}, a_{2k-1}{}^{(i)} t_k{}^{(i)} = a_{2k}{}^{(i)} t_{k+1}{}^{(i)} \ (i = 1, 2 ...., p; \ k = 1, 2, ...m - 1)$$

$$a_{2m-1}{}^{(i)} t_m{}^{(i)} = a_{2m}{}^{(i)}, y_m{}^{(i)} a_{2m}{}^{(i)} = d_i \qquad (5)$$

where $a_j{}^{(i)} \in U$ $(i = 1, 2, ..., p; \ j = 0, 1, 2 ..., 2m)$ and $t_q{}^{(i)}, y_q{}^{(i)} \in S^1$ $(i = 1, 2, ..., p; q = 0, 1, ..., m)$, and further for each $d_i \in S \backslash U$, we can assume that $t_q{}^{(i)}, y_q{}^{(i)} \in S \backslash U$ (from the proof of the Lemma 5.2.3)

In the following, we shall make use of Lemma 5.2.2 without explicit mention. We put

$$\tilde{x} = (x_1, x_2, ...x_p)$$

In this notation, the identity (2) is simple $u(\tilde{x}) = v(\tilde{x})$. Put

$$\tilde{d} = (d_1, d_2, ...d_p)$$

$$
\begin{aligned}
\tilde{a_k} &= (a_k{}^{(1)}, a_k{}^{(2)}, ..., a_k{}^{(p)}) \ (k = 0, 1, 2 ...., 2m) \qquad (6) \\
\tilde{t_q} &= (t_q{}^{(1)}, t_q{}^{(2)}, ..., t_q{}^{(p)}) \ (q = 0, 1, 2 ...., m) \\
\tilde{y_q} &= (y_q{}^{(1)}, y_q{}^{(2)}, ..., y_q{}^{(p)}) \ (q = 0, 1, 2 ...., m).
\end{aligned}
$$

We wish to show that $u(\tilde{d}) = v(\tilde{d})$. Now $\tilde{d} \in S^p$ is in the dominion of U^p in $(S^1)^p$ (where T^r, for any semigroup T and any integer $r \geq 2$, denotes the cartesian product of r-copies of the semigroup T) as $\tilde{d}$ has the following zigzag of length m

$$\tilde{d} = \tilde{a}_0\tilde{t}_1, \quad \tilde{a}_0 = \tilde{y}_1\tilde{a}_1,$$

$$\tilde{y}_k\tilde{a}_{2k} = \tilde{y}_{k+1}\tilde{a}_{2k+1}, \quad \tilde{a}_{2k-1}\tilde{t}_k = \tilde{a}_{2k}\tilde{t}_{k+1}, (k = 1, 2, 3......, m-1), \tag{7}$$

$$\tilde{a}_{2m-1}\tilde{t}_m = \tilde{a}_{2m}, \quad 3\tilde{y}_m\tilde{a}_{2m} = \tilde{d},$$

where $\tilde{a}_t \in U^p$ $(t = 0, 1, 2....., 2m)$ and $\tilde{a}_t, \tilde{t}_q \in (S^1)^p$ $(q = 1, 2, ..., m)$.

LEMMA 5.2.4: Let the word v in (2) begins with x_j, say if $d_j \in S\backslash U$, then

$$u(\tilde{d}\,) = v(\,\tilde{d}\,)$$

PROOF:

$$\begin{aligned}
u(\tilde{d}) &= u(\tilde{a}_0\ \tilde{t}_1\ ) \text{ (from equations (6))} \\
&= u(\tilde{a}_0\ )u(\tilde{t}_1\ ) \ \text{ (by Lemma (5.2.2) since each} a_0^{(i)} = y_1^{(i)}a_1^{(i)}) \\
&= v(\ \tilde{y}_1\ \tilde{a}_1\ )u(\tilde{t}_1\ ) \text{ (since } U \text{ satisfies(2))} \\
&= v(\tilde{y}_1\ )\ v(\tilde{a}_1\ )u(\tilde{t}_1) \text{ (by Corollary 4.1.4, since} y_1(i) \in S\backslash U) \\
&= v(\tilde{y}_1\ )\ u(\tilde{a}_1\ )u(\tilde{t}_1) \text{ (since } U \text{ satisfies (2))} \\
&= v(\tilde{y}_1)\ u(\tilde{a}_1\ \tilde{t}_1) \text{ (by Corollary 4.1.4, since } y_1^{(i)} \in S\backslash U) \\
&= v(\tilde{y}_1)\ u(\tilde{a}_2\ \tilde{t}_2) \text{ (from equations (6))} \\
&= \ldots\ldots \\
&= v(\tilde{y}_{m-1})\ u(\tilde{a}_{2m-2}\ \tilde{t}_m) \\
&= v(\tilde{y}_{m-1})\ u(\tilde{a}_{2m-2}\ \tilde{t}_m) \\
&= v(\tilde{y}_{m-1})\ u(\tilde{a}_{2m-2})\ u(\tilde{t}_m) \ \text{ (by Corollary 4.1.4, since } y_{m-1}^{(i)} \in S\backslash U)
\end{aligned}$$

$= \; v(\widetilde{y}_{m-1}) \; v(\widetilde{a}_{2m-2}) \; u(\widetilde{t}_m)$ (since U satisfies (2))

$= \; v(\widetilde{y}_{m-1} \; \widetilde{a}_{2m-2}) \; u(\widetilde{t}_m)$ (by Corollary 4.1.4, since $y_{m-1}^{(i)} \in S\backslash U$)

$= \; v(\widetilde{y}_m \; \widetilde{a}_{2m-1}) \; u(\widetilde{t}_m)$ (from eqations (6))

$= \; v(\widetilde{y_m})v(\widetilde{a}_{2m-1}) \; u(\widetilde{t}_m)$ (by Corollary 4.1.4, since $y_m^{(i)} \in S\backslash U$)

$= \; v(\widetilde{y}_m) \; u(\widetilde{a}_{2m-1} \; \widetilde{t}_m)$ (by Coro. 4.1.2, since $y_m^{(i)} \in S\backslash U$ and since U satisfies(2))

$= \; v(\widetilde{y}_m) \; u(\widetilde{a}_{2m})$ (from equations (6))

$= \; v(\widetilde{y}_m) \; v(\widetilde{a}_{2m})$ (since U satisfies (2))

$= \; v(\widetilde{y}_m \widetilde{a}_{2m})$ (by corollary 4.1.2 since $y_m(i) \in S\backslash U$,

$= \; v(\widetilde{d})$ (from equations (6))

This completes the proof of the Lemma 5.2.4.

we now return to the proof of Theorem 5.2.1, we regard the variables $x_1, x_2 \ldots x_n$ as being *replaced by* $d_1, d_2, \ldots d_n$ respectively, and it will be convenient for us to use the phrase *replaced by* in our proof. If all the variables in u and v are replaced from U then $u(\widetilde{d}) = v(\widetilde{d})$ as required. Hence we assume that in v, say, not every variable is replaced from U.

By Lemma 5.2.3, if the first variable of v is replaced by an element of $S\backslash U$, then we have the required result again. Hence we now consider the case where further the first variable of v is replaced by an element of U. Then,

$$v(\widetilde{x}) \; = \; v_1(\widetilde{x}) \; v_2(\widetilde{x}) \tag{7}$$

for some words v_1 and v_2 in the variables $x_1, x_2 \ldots x_p$ where v_1 is of the maximum length such that all the variables of v_1 are replaced by elements of U (the word v_1 is non empty and not all the variables $x_1, x_2, \ldots x_n$ appear in v_2) let the first variable of $v_2(\widetilde{x})$ be x_1, say (that is, x_1 is the first variable appearing in $v(\widetilde{x})$ which is replaced by the element of $S\backslash U$). For any i, if $d_i \in S\backslash U$, then $y_j^{(i)} \in S\backslash U$ for $j = 1, 2, \ldots, m$. Therefore, by

Results 1.2.5 and 1.2.9 for $d_i \in S\backslash U$,we can write

$$y_j^{(i)} = b_j^{(i)}\bar{y}_1^{(i)} \ and\ b_j^{(i)} = z_i^{(i)}c_j^{(i)} for j = 1,2,...,m \tag{8}$$

for some $b_j^{(i)}, c_j^{(i)}, \bar{y}_j^{(i)}, z_i^{(i)} \in S\backslash U$. For each $d_i \in U$, we put

$$b_j^{(i)} = c_j^{(i)} = \bar{y}_j^{(i)} = z_i^{(i)} = 1 \tag{9}$$

In addition to the notations (5) we shall also use the following:

$$\begin{aligned} \widetilde{b}_q &= (b_q^{(1)}, b_q^{(2)}, ...b_q^{(p)}) \ (q = 1,2,.....,m) \\ \bar{y}_q &= (\bar{y}_q^{(1)}\bar{y}_q^{(2)}, ..., \bar{y}_q^{(p)}) \ (q = 1,2,.....,m) \\ \widetilde{c}_q &= (c_q^{(1)}, c_q^{(2)}, ...c_q^{(p)})(q = 1,2,.....,m) \\ \widetilde{z}_q &= (z_q^{(1)}, z_q^{(2)}, ...z_q^{(p)}) \ (q = 1,2,.....,m) \end{aligned} \tag{10}$$

Now from equations (4) and (9), we have

$$\widetilde{y}_q = \widetilde{b}_q \, \widetilde{y}_q = \widetilde{z}_q \, \widetilde{c}_q \, \widetilde{y}_q \tag{11}$$

$$\begin{aligned} \text{Now } u(\widetilde{d}) &= u(\widetilde{a}_0\widetilde{t}_1) \ \text{(from equations (6))} \\ &= u(\widetilde{a}_0)u(\widetilde{t}_1) \ \text{(by Lemma 5.2.2, since } a_0^{(i)} = y_1^{(i)}a_1^{(i)} \text{ for i = 1,2,....,p)} \\ &= v(\widetilde{a}_0)u(\widetilde{t}_1) \ \text{(since } U \text{ satisfies (2))} \\ &= v(\widetilde{y}_1\widetilde{a}_1)u(\widetilde{t}_1) \ \text{(from equations (6))} \\ &= .... \\ &= v(\widetilde{y}_i\widetilde{a}_{2i-1})u(\widetilde{t}_i) \end{aligned}$$

(This is essentially an inductive assumption we now obtain equality with)

$$\begin{aligned} & v(\widetilde{y}_{i+1} \ \widetilde{a}_{2i+1})u(\widetilde{t}_{i+1}) = \\ = \ & v_1(\widetilde{y}_i \ \widetilde{a}_{2i-1})v_2(\widetilde{y}_i \ \widetilde{a}_{2i-1})u(\widetilde{t}_i) \ \text{(from equations (7))} \\ = \ & v_1(\widetilde{a}_{2i-1})v_2(\widetilde{y}_i \ \widetilde{a}_{2i-1})u(\widetilde{t}_i) \ \text{(since all variables of } v_1 \text{ are replaced from} U) \\ = \ & v_1(\widetilde{a}_{2i-1})v_2(\widetilde{y}_i) \ v_2(\widetilde{a}_{2i-1})u(\widetilde{t}_i) \ \text{(by corollary 4.1.4 since } v_2(\widetilde{y}_i \ \widetilde{a}_{2i-1}) \text{ begins with} \\ & a_{i-1}^{(l)} \text{ and } y_i^{(l)} \in S\backslash U) \\ = \ & v_1(\widetilde{a}_{2i-1})v_2(\widetilde{b}_i\widetilde{y}_i) \ v_2(\widetilde{a}_{2i-1})u(\widetilde{t}_i) \ \text{(from equations (11))} \end{aligned}$$

$= v_1(\widetilde{a}_{2i-1})v_2(\widetilde{b}_i)v_2(\widetilde{y}_i)\ v_2(\widetilde{a}_{2i-1})u(\widetilde{t}_i)$ (by corollary 4.1.4 since, $b_i^{(l)} = z_i^{(l)}c_i^{(l)}$ and $z_i^{(l)} \in S\backslash U$ from equations(8))

$= v_1(\widetilde{a}_{2i-1})v_2(\widetilde{b}_i\widetilde{a}_{2i-1})u(\widetilde{t}_i)v_2(\widetilde{y}_i)$ (by corollary 4.1.4 since, $b_i^{(l)} = z_i^{(l)}c_i^{(l)}$ and $z_i^{(l)} \in S\backslash U$ from equations(8))

$= v_1(\widetilde{b}_i\widetilde{a}_{2i-1})v_2(\widetilde{b}_i\widetilde{a}_{2i-1})u(\widetilde{t}_i)v_2(\widetilde{y}_i)$ (since all variables of v_1 are replaced from U)

$= v(\widetilde{b}_i\ \widetilde{a}_{2i-1})u(\widetilde{t}_i)\ v_2(\widetilde{y}_i)$ (from equations (7))

$= u(\widetilde{b}_i\ \widetilde{a}_{2i-1})u(\widetilde{t}_i)\ v_2(\widetilde{y}_i)$ (since U satisfies (2))

$= u(\widetilde{b}_i\ \widetilde{a}_{2i-1}\widetilde{t}_i)\ v_2(\widetilde{y}_i)$ (by Lemma 5.2.2, since if any $t_i^{(i)} \in S$ for any j, then $b_i^{(i)} = z_i^{(j)}c_i^{(j)}$ with $z_i^{(j)} \in S\backslash U$ from equation (8))

$= u(\widetilde{b}_i\ \widetilde{a}_{2i}\widetilde{t}_{i+1})\ v_2(\widetilde{y}_i)$ (from equations (6))

$= u(\widetilde{b}_i\ \widetilde{a}_{2i})u(\widetilde{t}_{i+1})\ v_2(\widetilde{y}_i)$ (by Lemma 5.2.2 since if any $t_{i+1}^{(j)} \in S\backslash U$ for any j, then $b_i^{(i)} = z_i^{(j)}c_i^{(j)}$ with $z_i^{(j)} \in S\backslash U$ from equation (8))

$= v(\widetilde{b}_i\ \widetilde{a}_{2i})u(\widetilde{t}_{i+1})\ v_2(\widetilde{y}_i)$ (since U satisfies (2))

$= v_1(\widetilde{b}_i\ \widetilde{a}_{2i})v_2(\widetilde{b}_i\ \widetilde{a}_{2i})u(\widetilde{t}_{i+1})\ v_2(\widetilde{y}_i)$ (from equations (7))

$= v_1(\widetilde{a}_{2i})v_2(\widetilde{b}_i\ \widetilde{a}_{2i})u(\widetilde{t}_{i+1})\ v_2(\widetilde{y}_i)$ (since all variables of v_1 are replaced from U)

$= v_1(\widetilde{a}_{2i})v_2(\widetilde{b}_i)\ v_2(\widetilde{y}_i)v_2(\widetilde{a}_{2i})u(\widetilde{t}_{i+1})$ (by Corollary 4.1.4, since $b_i^{(l)} = z_i^{(l)}c_i^{(l)}$ and $z_i^{(l)} \in S\backslash U$ from equation (8))

$= v_1(\widetilde{a}_{2i})v_2(\widetilde{b}_i\widetilde{y}_i\widetilde{a}_{2i})u(\widetilde{t}_{i+1})$ (by Corollary4.1.4, since $b_i^{(l)} = z_i^{(l)}c_i^{(l)}$ and $z_i^{(l)} \in S\backslash U$ from equation (8))

$= v_1(\widetilde{a}_{2i})v_2(\widetilde{y}_i\widetilde{a}_{2i})u(\widetilde{t}_{i+1})$ (from equation (11))

$= v_1(\widetilde{y}_i\widetilde{a}_{2i})v_2(\widetilde{y}_i\widetilde{a}_{2i})u(\widetilde{t}_{i+1})$ (since all variables of v_1 are replaced from U)

$= v(\widetilde{y}_i\widetilde{a}_{2i})u(\widetilde{t}_{i+1})$ (from equation (7))

$$
\begin{aligned}
&= v(\tilde{y}_{i+1}\tilde{a}_{2i+1})u(\tilde{t}_{i+1}) \text{ (if } i \leq m-1) \\
&= \ldots\ldots \\
&= v(\tilde{y}_m\tilde{a}_{2m}) \\
&= v(\tilde{d}),
\end{aligned}
$$

as required. This completes the proof of the Theorem 5.2.1.

A restatement of Theorem 5.2.1 in terms of permutative varieties gives us a generalization of [45, Theorem 2.3.1] which state that all commutative varieties are closed under epis.

THEOREM 5.2.5: Let v be a permutative variety defined by a permutation identity

$$x_1x_2\cdots x_n = x_{i_1}x_{i_2}\cdots x_{i_n}$$

such that $i_n \neq n(i_1 \neq 1)$. Then all subvarieties of the variety v are closed under epis.

Recall that an identity $u = v$ is epimorphically stable or stable under epis if all identities in conjunction with it are preserved under epis, by which we mean that if U is any semigroup satisfying $u = v$ and S is any epimorphic extension of U, then S satisfies all the identities satisfied by U.

In [30], P.M. Higgins has provided an example showing that some permutation identities are not epimorphically stable, namely those permutation identities which are consequences of the normality identity $xyzw = xzyw$. Theorem 5.2.5 gives a sufficient condition for permutation identities to be epimorphically stable. So as a joint result, in the following theorem, we determine all the permutation identities which are epimorphically stable.

THEOREM 5.2.6: Let v be a permutative variety defined by a permutation identity

$$x_1x_2\cdots x_n = x_{i_1}x_{i_2}\cdots x_{i_n}$$

is epimorphically stable and only if $i_n \neq n[i_1 \neq 1]$.

BIBLIOGRAPHY

[1] Aizenst, A.J., **Permutation Identities**, Modern Algebra, No. 3, 3-12, Leningrad Gos. Ped. Inst., Leningrad 1975.

[2] Bogdanovic, S. and Ciric, M., **A nil-extensionof a completely simple semigroup,** Publ. Inst. Math. 36(50) (1984), 45-50.

[3] Bogdanovic, S. and Ciric, M., **A note on left regular semigroups,** Publ. Math. Debrecen 48(1996), no. 3-4, 285-291.

[4] Burgess, W., **The meaning of mono and epi in some familiar categories,** Canad. Math. Bull. 8, No 6 (1965).

[5] Bulazewaska, A. and Krempa, J., **On epimorphisms in the category of all associative rings,** Bull. Acad. Polon. Sci. Ser. Sci. Maths. Astrom. Phys. Vol. 23, No. 11 (1975), 1153-1159.

[6] Clifford, A.H. and Preston, G.B., **The algebric theory of semigroups, Vol. I** Math. Soc., Providence, Rhode Island, 1961.

[7] Clifford, A.H. and Preston, G.B., **The algebric theory of semigroups, Vol. II** Math. soc., Providence, Rhode Island, 1967.

[8] Clarke, G.,**Semigroup varieties of inflations of union of groups,** Semigroup Forum, Vol. 23 (1981), 311-319.

[9] Chrislock, J.L., **A certain class of identities on semigroups,** Proc. Amer. Math. Soc. 21(1969), 189-190.

[10] Chon P.M., **Universal algebra,** Harper and Row, New York, 1965.

[11] Drbohlav, K., **A note on epimorphisms in algebraic categories,** Comment. Math. Uni. Caroline 4,2 (1863), 81-85.

[12] Dean, R. A. and Evans, T., **A remark on varieties of lattices and semigroups,** Proc. Amer. Math. Soc. 21 (1969), 394-396.

[13] Evans, T., **An embedding theorem for semigroups,** American J. Math. 76, (1954), 399-413.

[14] Evans, T., **The number of semigroup varieties,** Oxford Quart. J. (2), (1968).

[15] Evans, T., **The lattice of semigroup varieties,** Semigroup Forum Vol. 2 (1971), 1-43.

[16] Gardner, B.J., **Some aspects of T-nilpotence,** Pacific J. Math. Vol. 53, No. 1 (1974), 117-130.

[17] Gardner, B.J., **Epimorphisms of regular rings,** Comment. Math. Uni. Caroline 16 1(1975),151-160.

[18] Gardner, B.J., **A note on ring epimorphisms and polynomial identities,** Comment. Math. Uni. Carolinae 20, 2 (1979), 293-307.

[19] Gratzer, G., **Universal Algebra,** 2nd Ed. Springer-Verlag,

[20] Hall, T.E. , **On regular semigroups,** J. Algebra 24 (1973), 1-24.

[21] Hall, T.E., **Inverse semigroup varieties with the amalgamation property,** Semigroup Forum, Vol. 16 (1978), 37-51.

[22] Hall, T.E., **Represention extension and amalgamation for semigroups,** Quart. J. Math. Oxford (2), 29 (1978), 309-304.

[23] Hall, T.E., **Generalised inverse semigroups and amalgamation,** 'Semigroups' proceedings of the Monash Algebra Conference (1979), Acamedic Press: (London) 1980, 145-159.

[24] Hall, T.E. and Munn, W.D., **Semigroups satisfying minimal conditions II,** Glasgow Math. J. 20 (1979), 133-140.

[25] Hall, T.E., **Epimorphisms and dominions,** Semigroup Forum Vol. 24 (1982), 271-283.

[26] Hall, T.E. and Jones, P.R., **Epis are onto for finite regular semigroups,** Proc. Edinburgh Math. Soc. (20 26 (1983), no. 2, 151-162.

[27] Higgins, P.M., **Epis are onto for generalised inverse semigroups,** Semigroup Forum Vol. 23 (1981), 255-259.

[28] Higgins, P.M., **The commutative varieties of semigroups for epis are onto,** Proc. Edinburgh Math. Soc. Sect. A 94 (1983), no. 1-2, 1-7.

[29] Higgins, P.M., **The determination of all varietes consisting of absolutely closed semigroups,** Proc. Amer. Math. Soc. 87 (1983), no. 3, 419-421.

[30] Higgins, P.M., **Epimorphisms and semigroup varieties,** Ph.D. Thesis, Monash University, Australia, 1983.

[31] Higgins, P.M., **A semigroup with an epimorphically embedded subband,** Bull. Austral. Math. Soc. 27 (1983), 231-242.

[32] Higgins, P.M., **Epimorphisms, permutation identities and finite semigroups,** Semigroup Forum, Vol. 29 (1984), no. 1-2, 87-97.

[33] Higgins, P.M., **Saturated and epimorphically closed varieties of semigroups,** J. Aust. Math. Soc. Ser. A 36 (1984), no. 2, 153-175.

[34] Higgins, P.M., **Epimorphisms, dominions and semigroups,** Algebra Universalis 21 (1985), p.p. 225-233.

[35] Higgins, P.M., **Completely semisimple semigroup and epimorphisms,** Proc. Amer. Math. Soc. 96 (1986), p.p. 387-390.

[36] Higgins, P.M., **Dense subsets some common classes of semigroups,** Semigroup Forum Vol. 34 (1986), p.p. 5-19.

[37] Higgins, P.M., **Epimorphisms and amalgams,** Colleq. Math., Vol. 56 (1988), no. 1, 1-17.

[38] Howie, J.M., **An itroduction to semigroup theory,** London Math. Soc. Monographs 7. Academic Press, London, New York, San Francisco, 1976.

[39] Howie, J.M., **Epimorphisms and amalgamations,** A survey of recent progress. Semigroups (Szeged, 1981), 63-82 Colleq. Math. Soc. Janos. Bolyai, 39, North Holland Ansterdam, New York, 1985.

[40] Howie, J.M. and Isbell, J.R., **Epimorphisms and dominions II,** J. Algebra 6 (1967), 7-21.

[41] Imaoka, T., **Free products with amalgamation of bands,** Mem. Fac. Lit. & Sci., Shimane Univ. Nat. Sci., 10 (1976). 7-17.

[42] Isbell, **Epimorphisms and dominions,** Proceedings of the Conference on Categorical Algebra, La Jolla, 1965, 232-246 (Springer-Verlag, Berlin, New York, 1966).

[43] Khan, N.M. **Epimorphisms dominions and varieties of semigroups,** Semigroup Forum Vol. 25 (1982), 331-337.

[44] Khan,N.M. **Epimorphisms, Dominions and varieties of semigroups,** Ph.D. Thesis, Monash Univeristy, Australia, 1983.

[45] Khan, N.M., **Some saturated varieties of semigroups,** Bull. Austral. Math. Soc. Vol. 27 (111983), 419-425.

[46] Khan, N.M., **On saturated permutative varieties and consequences permutation identities,** J. Aust. Math. Soc. Ser. A. 38 (1985), 2, 186-197.

[47] Khan, N.M., **Epimorphically closed permutative varieties,** Trans. American Math. Soc. 287 (1985), no. 2, 507-528.

[48] Khan, N.M., **Homotypical identities and saturated varieties,** Proceedings of the international conference on ALGEBRA AND ITS APPLICATIONS held at A.M.U., Aligarh, India (Nov. 13-16, 1997).

[49] Khan, N.M., **On epimorphically closed homotypical permutative varieties,** to appear.

[50] Khan, N.M., **On epimorphisms, homotypical idendties and permutative varieties,** to appear.

[51] KIsielewiez, Andrzeg, **Varieties of commutative semigroups,** Trans. Amer. Math. Soc. 27 (1983), no. 3, 419-425.

[52] KIsielewiez, Andrzeg, **Varieties of commutative semigroups,** Trans. Amer. Math. Soc. 342 (1984), no.1, 275-306.

[53] Lee, S.M.,**Rings and semigroups which satisfy the identity** $(xy)^n = xy =$ Nanta Math. 6 (1973),no. 1, 21-28.

[54] Mitchell, B., **Theory of categories,** Academic Press, 1965.

[55] Mead, D.G. and Tamura, T., **Semigroups satisfying** $xy^m = yx^m = (xy^m)^m$, Proc. Japan Acad. 44 (1968), 779-781.

[56] Munn, W.D., **Semigroups satisfying minimal conditions,** Proc. Glasgow Math. Assoc. 3 (1957), 145-152.

[57] Munn, W.D., and Penrose R., **A note on Inverse semigroups,** Proc. Camb. Phil. Soc. 51 (1955), 396-399.

[58] Nordhal, T., **Semigroup satisfying** $(xy)^m = x^m y^m$, Semigroup Forum Vol. 8 (1974), 332-346.

[59] Petrich, M., **Introduction to semigroups,** Merill, Ohio, Columbus, 1973.

[60] Petrich, M., **Rings and semigroups,** Lecture Notes in Math., Vol. 380, Springer - Variag, Berlin, 1974.

[61] Petrich, M., **The structure of completely regular semigroups,** Trans. Amer. Math. Soc. 189 (1974), 211-236.

[62] Petrich, M., **Inverse semigroups,** J. Willy & Sons, New York, 1984.

[63] Perkins, P. **Bases for equational theories of semigroups,** J. Algebra, 11 (1969), 298-314.

[64] Pollak, G., **On the consequences permutation identities,** Acta. Sci. Math. 1973 (34), 323-333.

[65] Putcha, M.S. Yaqub, A., **Semigroups satisfying permutation identities,** Semigroup Forum Vol. 3 (1971), 68-73.

[66] Salii, V.N., **Equational complete varities of semigroups,** Izv Vysh Uchebn, Zaved Mat. 5 (1969), 61-68.

[67] Sapir, M.V. and Suhanov, E.V., **ON varities of periodic semigroups,** Izv. Vyzor. Mat. 4 (1981), 48-55 (In Russian).

[68] Scheiblich, H.E., **On epis and dominions of bands,** Semigroup Forum Vol. 13 (1976), 103-114.

[69] Scheiblich, H.E. and Moore, K.C., T_x **is absolutely closed,** Semigroup Forum Vol. 6 (1973), 216-226.

[70] Shevrin, L.N. and Valkov M.V., **Identities on semigroups,** Izv. Vysh. Uchebn. Zav. Mat. 11 (1985),3-47.

[71] Tamura, T., **Nil orders of commutative nil-bounded semigroups,** Semigroup Forum, Vol. 24 (1982), 255-262.

[72] Tamura, T., **The theory of construction of finite semigroups III**, Finite unipotent semigroups, Osaka Math. J. 10 (1958), 191-204.